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ON COEFFICIENT PROPERTIES OF POWER SERIES

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Summary. In this paper we consider some applications of the function spaces $B^s_{p,q,+}$ and $F^s_{p,q,+}(0<p,\ q\leqq\infty,\ -\infty<s<\infty)$, recently investigated by the author [9] to coefficient estimates of power series and obtain various known and new results in a straightforward manner.

Section 1 contains the definition of the spaces under consideration and a short summary of their basic properties. In Section 2 we deal with coefficients and lacunary coefficients of power series.

1. Let $f=f(e^{it})=\Sigma_{n=0}^{\infty}c_n e^{int}\in D'_+,\ t\in(-\pi,\pi]$, be a periodic distribution of power series type and $f(z)=\Sigma_{n=0}^{\infty}c_n Z^n$ the corresponding analytic function in the unit disc $(|z|<1)$. By definition f belongs to the Hardy space H_p $(0<p<\infty)$, if

$$\|f\|_{H_p}=\sup_{r<1}\{(2\pi)^{-1}\int_{-\pi}^{\pi}|f(re^{it})|^p dt\}^{1/p}<\infty.$$

For any given sequence $\Lambda=\{\lambda^{(j)}\}$, $j\geqq 0$, of coefficient multipliers $\lambda^{(j)}=\{\lambda_n^{(j)}\}$, $n\geqq 0$, with the properties

$$\text{supp}\,\lambda^{(j)}=\{n\geqq 0:\ \lambda_n^{(j)}\neq 0\}\subset\begin{cases}\{2^{j-1}+1,\dots,2^{j+1}\}, & j>0,\\ \{0,\ 1,\ 2\}, & j=0,\end{cases}$$

(1)

$$\sum_{j=0}^{\infty}\lambda_n^{(j)}=1,\quad n\geqq 0,$$

we define a variety of quasi-norms of Littlewood-Paley type by

$$\|f\|_{B^{s,\Lambda}_{p,q,+}}=\|\{2^{js}\cdot\lambda^{(j)}f\}\|_{l_q(L_p)}$$

$$=\{\sum_{j=0}^{\infty}2^{jsq}\cdot(\int_{-\pi}^{\pi}|\sum_{n=0}^{\infty}\lambda_n^{(j)}c_n e^{int}|^p dt)^{q/p}\}^{1/q}$$

and

$$\|f\|_{F^{s,\Lambda}_{p,q,+}}=\|\{2^{js}\cdot\lambda^{(j)}f\}\|_{L_p(l_q)}=\|\{\sum_{i=0}^{\infty}2^{jsq}\cdot|\sum_{n=0}^{\infty}\lambda_n^{(j)}c_n e^{int}|^q\}^{1/q}\|_{L_p}$$

(with obvious modifications if $p=\infty$ or $q=\infty$), respectively. Now we set

$$B^s_{p,q,+}=\{f\in D'_+ : \|f\|_{B^s_{p,q,+}}=\inf\|f\|_{B^{s,\Lambda}_{p,q,+}}<\infty\};$$

(2)

$$F^s_{p,q,+}=\{f\in D'_+ : \|f\|_{F^s_{p,q,+}}=\inf\|f\|_{F^{s,\Lambda}_{p,q,+}}<\infty\},$$

where the infimum is taken over all Λ, satisfying (1), and

(3) $$\|\Lambda\|_\alpha=\sup_j\|\lambda^{(j)}\|_{bv_{\alpha+1}}<\infty \text{ for some } \alpha>\alpha_0.$$

The sequence spaces $bv_{\alpha+1}$, $\alpha\geqq 0$, of generalized bounded variation are defined by

$$bv_{\alpha+1}=\{\{\lambda_n\}: \|\lambda\|_{bv_{\alpha+1}}=\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\cdot|\Delta^{\alpha+1}\lambda_k|+\lim_{n\to\infty}|\lambda_n|<\infty\},$$

for further information see [11]. In (2), (3) and in the following, we consider $0<p$, $q\leqq\infty$, $-\infty<s<\infty$, $\alpha_0=\max(0, 1/p-1)$ in the case of $B^s_{p,q,+}$ spaces and $0<p<\infty$, $0<q\leqq\infty$, $-\infty<s<\infty$, $\alpha_0=\max(0, 1/p-1, 1/q-1)$ in the case of $F^s_{p,q,+}$ spaces.

In [9] we proved the following properties of the spaces (2).

Theorem 1. *Let X be any of the spaces, defined in* (2). *Then X is a quasi Banach space with quasi-norm $\|\cdot\|_X$ (Banach space, if $p, q\geqq 1$) and for every Λ, satisfying* (1), (3), *the related $\|\cdot\|_{X^\Lambda}$ is an equivalent quasi-norm on X.*

Theorem 2. *Let X be any of the spaces, defined in (2). Then a sequence $\eta=\{\eta_i\}$ of complex numbers is a coefficient multiplier for X, if*

(4) $$\eta\in bv_{\alpha+1} \textit{ for some real } \alpha>\alpha_0, \textit{ or}$$

(5) $$\|\eta\|_{l_\infty}+\sup_{j>0}2^{mj}\cdot\sum_{i=2^j}^{2^{j+1}}|\Delta^{m+1}\eta_i|<\infty \textit{ for some integer } m>\alpha_0.$$

These are multiplier criteria of Mihlin-Hörmander-Marcinkiewicz type. The next theorem describes various equivalent quasi-norms.

Theorem 3. *The following representations hold with equivalent quasi-norms:*

(6) $$F^0_{p,2,+}=H_p,\quad 0<p<\infty,$$

(7) $$F^s_{p,q,+}=\{f\in D'_+ : \|f\|=\{\int_{-\pi}^{\pi}(\int_0^1(1-r)^{-q(\beta+s)-1}|J^\beta f(re^{it})|^q dr)^{p/q}dt\}^{1/p}<\infty\},$$

$$\beta+s<0,\ 0<p<\infty,\ 0<q\leqq\infty, \textit{ where } J^\beta f(e^{it})=\sum_{n=0}^{\infty}(n+1)^{-\beta}c_n e^{int},$$

(8) $$B^s_{p,q,+}=H\Lambda(s, p, q)^{(*)}=\{f\in D'_+ : \|f\|$$

$$=\{\int_0^1(1-r)^{-q(\beta+s)-1}(\int_{-\pi}^{\pi}|J^\beta f(re^{it})|^p dt)^{q/p}dr\}^{1/q}<\infty\},$$

$$\beta+s<0,\ 0<p,\ q\leqq\infty,$$

(*) See Flett [5].

(9) $$B^s_{p,q,+}=\{f\in H_p:\ \|f\|=\|f\|_{H_p}+\{\int_0^\pi t^{sq-1}\cdot\omega_m(f,t)^q_p dt\}^{1/q}<\infty\},$$
$$0<s<m,\ 0<p,\ q\leq\infty,\ m>0,$$

(10) $$B^s_{p,q,+}=\{f\in H_p:\ \|f\|=\|f\|_{H_p}+\{\sum_{k=0}^{\infty}2^{ksq}\cdot E_{2^k}(f)^q_p\}^{1/q}<\infty\},$$
$$0<s<\infty,\ 0<p,\ q\leq\infty,$$

(with obvious modifications, if $p=\infty$ or $q=\infty$), where $\omega_m(f,t)_p$ and $E_n(f)_p$ denote moduli of smoothness and best approximations by polynomials of f in H_p, respectively.

Furthermore, the operator J^β of fractional integration with real β yields an isomorphism of X^s onto $X^{s+\beta}$ for arbitrary p, q, s, and $X=B^s_{p,q,+}$ or $X=F^s_{p,q,+}$ (lift property). Duality and interpolation properties of the spaces (2) can be studied in analogy to the case of R^n (cf. [12]).

Theorem 4. *Let $-\infty<s<\infty$. Then we have (with the corresponding quasi-norm estimates) the imbeddings*

(11) $$B^s_{p_0,q,+}\hookrightarrow B^{s-1/p_0+1/p_1}_{p_1,q,+},\quad 0<p_0<p_1\leq\infty,\quad 0<q\leq\infty;$$

(12) $$F^s_{p_0,q_0,+}\hookrightarrow F^{s-1/p_0+1/p_1}_{p_1,q_1,+},\ 0<p_0<p_1<\infty,\quad 0<q_0,\ q_1\leq\infty;$$

(13) $$F^s_{p_0,q,+}\hookrightarrow B^{s-1/p_0+1/p_1}_{p_1,p_0,+},\quad 0<p_0<p_1\leq\infty,\quad 0<q\leq\infty;$$

(14) $$B^s_{p_0,p_1,+}\hookrightarrow F^{s-1/p_0+1/p_1}_{p_1,q,+},\quad 0<p_0<p_1<\infty,\quad 0<q\leq\infty.$$

The proof of these imbeddings is standard (see, for instance, [12, p. 100-104] and [4, 5, 8]), we omit it here.

2. In order to obtain coefficient estimates for f, belonging to some of the spaces described above, we shall use Theorem 4 and the obvious relation (here Σ' means that for $j=0$ we should set $\Sigma^1_{n=0}$)

(15) $$\|f\|_{B^s_{2,q,+}}\approx\begin{cases}\{\sum_{j=0}^{\infty}2^{jsq}(\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2)^{q/2}\}^{1/q}, & 0<q<\infty,\\ \sup_{j\geq 0}2^{js}(\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2)^{1/2}, & q=\infty.\end{cases}$$

From (11) and (15) we immediately get

Proposition 1. Let $0<q<\infty$, $-\infty<s<\infty$.

a) If $f\in B^s_{p,q,+}$ for some $0<p\leq 2$, then

(16) $$\{\sum_{j=0}^{\infty}2^{jq(s-1/p+1/2)}(\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2)^{q/2}\}^{1/q}\leq C\cdot\|f\|_{B^s_{p,q,+}}<\infty.$$

b) If for the coefficients of $f\in D'_+$ the term in the left-hand side of (16) is finite for some $2\leq p\leq\infty$, so we have $f\in B^s_{p,q,+}$, and the inverse inequality in (16) is valid. With obvious modifications the statement also holds for $q=\infty$.

Quasi-norms of the type used in Proposition 1 for coefficient sequences are of some interest in connection with multiplier criteria in H_p spaces (see [7]) and other problems.

We define the quasi-normed sequence spaces l_q^s by

$$l_q^s=\{\{c_n\}: \|c_n\|_{l_q^s}=\{\sum_{n=0}^{\infty}(n+1)^{sq}|c_n|^q\}^{1/q}<\infty\},\quad 0<q<\infty$$

(modification if $q=\infty$), $-\infty<s<\infty$.

Proposition 2. Let $0<q\leqq\infty$, $-\infty<s<\infty$.

a) If $f\in B^s_{p,q,+}$ with $0<p\leqq 2$ then $\{c_n\}\in l_q^{s-1/p-1/q'+1}$, i. e.

$$\|c_n\|_{l_q^{s-1/p-1/q'+1}}\leqq C\cdot\|f\|_{B^s_{p,q,+}},\quad q'=\min(2,q). \tag{17}$$

b) If $\{c_n\}\in l_q^{s-1/p-1/q''+1}$ with $2\leqq p\leqq\infty$ then $f\in B^s_{p,q,+}$, and

$$\|f\|_{B^s_{p,q,+}}\leqq C\cdot\|c_n\|_{l_q^{s-1/p-1/q''+1}},\quad q''=\max(2,q). \tag{18}$$

These inequalities are a direct consequence of Proposition 1 and

$$C\cdot 2^{j(1/q''-1/2)}\{\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2\}^{1/2}\leqq\{\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^q\}^{1/q}$$

$$\leqq C\{\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2\}^{1/2}\cdot 2^{j(1/q'-1/2)},\quad j\geqq 0,$$

$0<q<\infty$ (modification if $q=\infty$).

In the case of $F^s_{p,q,+}$ we use (13), (14) instead of (11).

Proposition 3. Let $0<q\leqq\infty$, $-\infty<s<\infty$.

a) If $0<p<2$ or $p=2$, $0<q\leqq 2$, then

$$\|c_n\|_{l_p^{s+1-2/p}}\leqq C\{\sum_{j=0}^{\infty}2^{jp(s-1/p+1/2)}(\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2)^{p/2}\}^{1/p} \tag{19}$$

$$\leqq C\cdot\|f\|_{F^s_{p,q,+}},\quad f\in F^s_{p,q,+}.$$

b) If $2<p<\infty$ or $p=2$, $2\leqq q\leqq\infty$, then

$$\|f\|_{F^s_{p,q,+}}\leqq C\{\sum_{j=0}^{\infty}2^{jp(s-1/p+1/2)}(\sum_{n=2^j}^{2^{j+1}-1}{}'|c_n|^2)^{p/2}\}^{1/p} \tag{20}$$

$$\leqq C\cdot\|c_n\|_{l_p^{s+1-2/p}}$$

and $f\in F^s_{p,q,+}$ under the assumption that one of the coefficient quasi-norms in (20) is finite, in particular if $\{c_n\}\in l_p^{s+1-2/p}$.

The proof is obvious (e. g., to obtain (19) take $p_0=p$, $p_1=2$ in (13) and apply (15)).

Propositions 1-3 are certain generalizations of classical Hardy-Littlewood results on Fourier coefficients of trigonometric and power series (cf. [6, 13]). To see this we consider some special cases. If we put $s=0$, $q=2$, it follows from (19), (20) and the representation (6) that

$$(21)\qquad \begin{aligned} &\{\sum_{n=0}^{\infty}|c_n|^p(n+1)^{p-2}\}^{1/p}\leq C\cdot\|f\|_{H_p}, \quad 0<p\leq 2;\\ &\|f\|_{H_p}\leq C\cdot\{\sum_{n=0}^{\infty}|c_n|^p(n+1)^{p-2}\}^{1/p}, \quad 2\leq p<\infty. \end{aligned}$$

This is a theorem of Hardy-Littlewood. A certain generalization of (21) due to H. R. Pitt

$$(22)\qquad \|c_n\|_{l_q^{1-1/p-1/q-s}}\leq C\cdot\|(1-z)^s f(z)\|_{H_p}, \quad 0<p\leq q\leq\infty,$$

where $s\geq\max(0, 1-1/p-1/q)$, can be obtained in a similar way, at least if $0<p\leq q\leq 2$. Of course, denoting $g(z)=(1-z)^s f(z)$, from (13) and (6) we have

$$g(z)\in H_p=F^0_{p,2,+}\subset B^{1/r-1/p}_{r,p,+}\subset B^{1/r-1/p}_{r,q,+}, \quad p\leq q,$$

where r is an appropriate chosen real ($p<r<2$). Thus, using (8), we get $f\in B^{1/r-1/p-s}_{r,q,+}$ (see [3, p. 751]) with corresponding quasi-norm estimates and Proposition 2 gives the required estimate

$$\|c_n\|_{l_q^{1-1/p-1/q-s}}\leq C\cdot\|f\|_{B^{1/r-1/p-s}_{r,q,+}}\leq C\cdot\|g\|_{H_p}, \quad 0<p\leq q\leq 2.$$

Another application of Proposition 2 yields some results of [1, 2], concerning the Banach spaces $B^p=H\Lambda(1-1/p, 1, 1)\cup H_p$, $0<p<1$, which coincide with $B^{1-1/p}_{1,1,+}$ (see (8)). E. g., the inequality

$$(23)\qquad \|c_n\|_{l_1^{-1/p}}=\{\sum_{n=0}^{\infty}|c_n|\cdot(n+1)^{-1/p}\}\leq C\cdot\|f\|_{B^p}, \quad f\in B^p,$$

stated in [2], follows from (17).

Finally, it should be added that the relations

$$(24)\qquad |c_n|=\left\{\begin{array}{ll} o(n^{-s+\max(1/p-1,\ 0)}), & 0<q<\infty\\ O(n^{-s+\max(1/p-1,\ 0)}), & q=\infty \end{array}\right\}\leq C\cdot\|f\|_{B^s_{p,q,+}}\cdot n^{-s+\max(1/p-1,\ 0)}$$

$n\geq 0$, $0<p\leq\infty$, $-\infty<s<\infty$, can easily be obtained from

$$(25)\qquad |c_n|\leq C\cdot\left|\int_{-\pi}^{\pi}\lambda^{(j)}f(re^{it})\cdot(re^{it})^{-n}dt\right|\leq C\cdot n^{\max(1/p-1,\ 0)}\|\lambda^{(j)}f\|_{L_p}$$

(to verify the latter inequality for $p<1$ take $r=r_n=1-1/n$ and use the Hardy-Littlewood inequality $\|f(re^{it})\|_{H_1}\leq C\cdot(1-r)^{1-1/p}\|f\|_{H_p}$, $f\in H_p$) and the definition (2) of the spaces $B^s_{p,q,+}$. The corresponding result for the spaces $F^s_{p,q,+}$ is a consequence of (24) and the imbeddings

$$(26)\qquad B^s_{p,\min(p,q),+}\subset F^s_{p,q,+}\subset B^s_{p,\max(p,q),+},$$

In the concluding part of this paper we briefly deal with lacunary power series and lacunary coefficients. An increasing sequence $\{n_k\}$ of integers is said to be lacunary, if $n_{k+1}/n_k \geq \delta > 1$, $k \geq 0$.

Proposition 4. Let $0<p, q\leqq\infty$ ($p<\infty$ in the case of $F^s_{p,q,+}$), and $-\infty<s<\infty$. Then for lacunary $\{n_k\}$ the power series $f=\Sigma_{k=0}^{\infty} c_{n_k} e^{in_k t}$ belongs to $B^s_{p,q,+}$ ($F^s_{p,q,+}$) if and only if $\{n_k^s c_{n_k}\} \in l_q$, more precisely, we have

$$(27)\qquad \{\sum_{k=0}^{\infty} n_k^{sq} |c_{n_k}|^q\}^{1/q} \approx \|f\|_{B^s_{p,q,+}} \quad (\approx \|f\|_{F^s_{p,q,+}}).$$

Proof. If $\delta>4$, then for any f under consideration we have $n_k \in \{2^{j-1}, \dots, 2^{j+1}\}$ for at most one $k \geqq 0$. Thus,

$$2^{js} \cdot \lambda^{(j)} f(e^{it}) = \begin{cases} 0, & n_k \notin \{2^{j-1}, \dots, 2^{j+1}\}, \\ 2^{js} \cdot \lambda^{(j)}_{n_k} \cdot c_{n_k} \cdot e^{in_k t}, & n_k \in \{2^{j-1}, \dots, 2^{j+1}\}, \end{cases}$$

$j \geqq 0$, and (27) immediately follows from the Definition (2). Observing that every lacunary sequence $\{n_k\}$ can be decomposed into a finite collection of lacunary sequences with $\delta>4$, this yields Proposition 4 in complete generality.

Proposition 5. Let $0<p$, $q\leqq\infty$, $-\infty<s<\infty$, $\gamma=\max(0, 1/p-1)$. Then we have for any lacunary sequence $\{n_k\}$

$$(28)\qquad \|\{n_k^{s-\gamma} \cdot c_{n_k}\}_{l^q} \leqq C \cdot \|f\|_{B^s_{p,q,+}}.$$

The proof of (28) is a consequence of (25) and (2).

According to (6), (28) and the imbeddings (13) ($0<p<1$) or (26) ($p=1$) as a corollary for $f \in H_p$ we obtain the estimates

$$(29)\qquad \|f\|_{H_p} \geqq C \begin{cases} \|\{n_k^{1-1/p} c_{n_k}\}\|_{l_p}, & 0<p<1, \\ \|c_{n_k}\|_{l_2}, & p=1. \end{cases}$$

The case $p=1$ represents the well-known Paley theorem on lacunary coefficients in H_1 (cf. [13, v. 2, p. 133]), while $0<p<1$ was considered in [2].

Finally, it should be mentioned that most of the estimates, obtained in this section, are the best ones in the corresponding scales of quasi-norms. On the other hand, using the definitions of the spaces $B^s_{p,q,+}$ and $F^s_{p,q,+}$ (together with Theorem 3), some inequalities, inverse to those considered above, can also be established. For instance, if $f=\Sigma_{n=0}^{\infty} c_n e^{int} \in D'_+$ and $0<p$, $q\leqq 2$, $-\infty<s<\infty$, then we obtain from (2), (15)

$$\|f\|_{B^s_{p,q,+}} \leqq \|f\|_{B^s_{2,q,+}} \leqq C\{\sum_{j=0}^{\infty} 2^{jsq} (\sum_{n=2^j}^{2^{j+1}-1}{}' |c_n|^2)^{q/2}\}^{1/q}.$$

Now, supposing $\{c_n\} \in l^s_q$ or $|c_n| = 0(n^\alpha)$, $n \to \infty$, for some $\alpha < -s-1/2$, this yields $f \in B^s_{p,q,+}$ (for the special case $B^p = B^{1-1/p}_{1,1,+}$, $0<p<1$, see [2, p. 260]).

N o t e a d d e d i n p r o o f. As it was pointed out to us by H. T r i e b e l some results analogous to Proposition 2 have been obtained by A. P i e t s c h in the more general framework of approximation spaces. (cf. A. P i e t s c h. Approximation spaces. [Analogies between spaces of sequences, functions and operators]. *J. Approx. Theory,* **32,** 1981, No 2. 115-134.)

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