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ON THE WALSH-FOURIER TRANSFORM OF FUNCTIONS IN $L^2(R^+)$

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Summary. In the paper [3] we constructed a complete orthonormal system in $L^2(R^+)$, which consists of eigenfunctions of the Walsh-Fourier transform. In this article we define the Walsh-Fourier transform of functions in $L^2(R^+)$, using these eigenfunctions, and give an explicit formula for this transform.

Let us recall some definitions in connection with the Walsh-Fourier transform. Let $x \in R^+$ be an arbitrary number. Then x has (if it is not dyadically rational number) a uniquely determined dyadic expansion $x = \Sigma_{i=-\infty}^{\infty} 2^{-i-1} x_i$, where $x_i \in \{0, 1\}$ $(i \in Z)$ and there is a $\nu_x \in Z$ such that $x_i = 0$ $(i < \nu_x)$. If x is a dyadically rational number, then we consider the finite expansion of x (i. e. there is a $\mu_x \in Z$ such that $x_i = 0$ $(i > \mu_x)$). If $y \in R^+$ has the dyadic expansion $y = \Sigma_{i=-\infty}^{\infty} 2^{-i-1} y_i$, then we define

$$x \overset{\circ}{+} y := \sum_{i=-\infty}^{\infty} 2^{-i-1} (x_i + y_i) \bmod 2\,; \quad x \circ y := \sum_{i=-\infty}^{\infty} 2^{-i-1} z_i,$$

where $z_i := \Sigma_{k+l=i-1} x_k y_l \pmod 2$ $(i \in Z)$. With these operations the set R^+ is a field, the so-called dyadic field (see [1]). If we define $\rho(x, y) := x \overset{\circ}{+} y$ $(x, y \in R^+)$, then the dyadic field is a locally compact topological field with respect to the topology induced by the metric ρ. The characters of the additive group of the dyadic field are the mappings

$$\psi_y : R^+ \to T, \ \psi_y(x) := (-1)^{\Sigma_{i=-\infty}^{\infty} x_i y_{-1-i}},$$

where T is the multiplicative group of the complex numbers with absolute value 1 and $y \in R^+$ is an arbitrarily fixed number. The functions ψ_y $(y \in R^+)$ are called the generalized Walsh functions (see [1]). It can be shown that

$$\text{(1)} \qquad \begin{aligned} &\text{i)} \ \psi_y(x) = \psi_x(y) \\ &\text{ii)} \ \psi_y(x) = \psi_{[y]}(x)\, \psi_{[x]}(y), \end{aligned} \qquad (x, y \in R^+)$$

where [ ] denotes the integer part and $\{\psi_n : n \in N\}$ is the Walsh-Paley system.

Let $f \in L^1(R^+)$ be an arbitrary Lebesgue integrable function. The Walsh-Fourier transform of the function f is defined by the formula

(2) $$\widehat{f}(y):=\int_{R^+} f(x)\,\psi_y(x)\,dx \quad (y\in R^+)$$

(see [1]).

Let us define the functions $\varphi_{k,n,+}$, $\varphi_{k,n,-}$ $(k, n\in N)$ in the following way:

(3)
$$\varphi_{k,n,+}(x):=\begin{cases} 2^{-1/2}\psi_{k+n}(x); & x\in[k,k+1),\\ 2^{-1/2}\psi_k(x)\;; & x\in[k+n,k+n+1),\\ 0\;; & \text{otherwise,}\end{cases}$$
$$\varphi_{k,n,-}(x):=\begin{cases} 2^{-1/2}\psi_{k+n}(x); & x\in[k,k+1),\\ -2^{-1/2}\psi_k(x); & x\in[k+n,k+n+1),\\ 0\;; & \text{otherwise.}\end{cases}$$

In the paper [3] it is shown that $\widehat{\varphi}_{k,n,+}=\varphi_{k,n,+}$, $\widehat{\varphi}_{k,n,-}=-\varphi_{k,n,-}$ $(k, n\in N)$ and $\Phi:=\{\varphi_{k,n,+}, \varphi_{k,n,-}: k, n\in N\}$ is a complete orthonormal system in $L^2(R^+)$.

Now let us consider a function $f\in L^2(R^+)$ and denote by $c_{k,n,+}(f)$, $c_{k,n,-}(f)$ $(k, n\in N)$ the Fourier coefficients of the function f with respect to the system Φ

(4)
$$c_{k,n,+}(f):=\langle f, \varphi_{k,n,+}\rangle=\int_{R^+} f\,\varphi_{k,n,+},$$
$$c_{k,n,-}(f):=\langle f, \varphi_{k,n,-}\rangle=\int_{R^+} f\,\varphi_{k,n,-}.$$

Then $f=\sum_{k,n\in N}(c_{k,n,+}(f)\varphi_{k,n,+}+c_{k,n,-}(f)\,\varphi_{k,n,-})$ in $L^2(R^+)$-norm. We can define the Walsh-Fourier transform of the function f by the following equality:

(5)
$$\mathscr{F}f:=\sum_{k,n\in N}(c_{k,n,+}(f)\,\widehat{\varphi}_{k,n,+}+c_{k,n,-}(f)\,\widehat{\varphi}_{k,n,-})$$
$$=\sum_{k,n\in N}(c_{k,n,+}(f)\varphi_{k,n,+}-c_{k,n,-}(f)\,\varphi_{k,n,-})$$

(since, by the Riesz-Fischer theorem, the series above converges in $L^2(R^+)$) With the use of this definition it is easily proved that the operator $\mathscr{F}: L^2(R^+)\to L^2(R^+)$ is a linear unitary operator of the Hilbert space $L^2(R^+)$ and $\mathscr{F}(\mathscr{F}f)=f\;(f\in L^2(R^+))$.

Now let us take the following order of the series (5) (see [4]):

$$\sum_{N=0}^{\infty}\sum_{k+n=N}(c_{k,n,+}(f)\,\varphi_{k,n,+}-c_{k,n,-}(f)\,\varphi_{k,n,-}).$$

Using (3) and (4), we have

$$c_{k,n,+}(f)\varphi_{k,n,+}(x)-c_{k,n,-}(f)\,\varphi_{k,n,-}(x)$$
$$=\begin{cases}\widehat{f}|_{[k+n,\,k+n+1)}(k)\,\psi_{k+n}(x); & x\in[k,k+1),\\ \widehat{f}|_{[k,\,k+1)}(k+n)\,\psi_k(x)\;; & x\in[k+n,k+n+1),\\ 0\;; & \text{otherwise,}\end{cases}$$

where $\widehat{f}|_{[l,l+1)}(j)$ $(l, j\in N)$ denotes the j-th Walsh-Fourier coefficient of the function $f|_{[l,l+1)}$. Moreover,

$$\sum_{k+n=N}(c_{k,n,+}(f)\,\varphi_{k,n,+}(x)-c_{k,n,-}(f)\,\varphi_{k,n,-}(x))$$

$$=\begin{cases}\widehat{f|_{[N,N+1)}}(j)\,\psi_N(x) & ;\quad x\in[j,j+1),\quad j=0,1,\dots,N-1,\\ \sum_{l=0}^{N}\widehat{f|_{[l,l+1)}}(N)\,\psi_l(x) & ;\quad x\in[N,N+1),\\ 0 & ;\quad \text{otherwise,}\end{cases}$$

and

$$\sum_{N=0}^{M-1}\sum_{k+n=N}(c_{k,n,+}(f)\,\varphi_{k,n,+}(x)-c_{k,n,-}(f)\,\varphi_{k,n,-}(x))$$

$$=\begin{cases}\sum_{l=0}^{M-1}\widehat{f|_{[l,l+1)}}(j)\,\psi_l(x) \;; & x\in[j,j+1),\ j=0,1,\dots,M-1,\\ 0 \;; & \text{otherwise.}\end{cases}$$

With the use of (1) ii) it is easy to verify that this equality can be written in the form

$$\sum_{N=0}^{M-1}\sum_{k+n=N}(c_{k,n,+}(f)\,\varphi_{k,n,+}(x)-c_{k,n,-}(f)\,\varphi_{k,n,-}(x))$$

$$=\begin{cases}\int_0^M f(t)\,\psi_x(t)\,dt\,; & x\in[0,M),\\ 0 \;; & \text{otherwise,}\end{cases}$$

i. e. we have for the Walsh-Fourier transform of the function $f\in L^2(\mathbf{R}^+)$

(6) $$\mathcal{F}f=\lim\,(F_M),$$

where

$$F_M:=(f\cdot\chi_{[0,M)})^{\widehat{}}\cdot\chi_{[0,M)}\quad(M\in\mathbf{N})$$

(χ_A denotes the characteristic function of the set $A\subset\mathbf{R}^+$).

We notice that equality (6) can be shown only using the unitary property of the operator $\mathcal{F}$.

In the following we give an explicit formula for the Walsh-Fourier transform $\mathcal{F}f$. Let $\xi\in\mathbf{R}^+$ be an arbitrary number. Then from (5) we have

$$\langle\mathcal{F}f,\chi_{[0,\xi]}\rangle=\sum_{k,n\in\mathbf{N}}(c_{k,n,+}(f)\,\langle\widehat{\varphi}_{k,n,+},\chi_{[0,\xi]}\rangle+c_{k,n,-}(f)\,\langle\widehat{\varphi}_{k,n,-},\chi_{[0,\xi]}\rangle),$$

i. e.

$$\int_0^\xi\mathcal{F}f(x)\,dx=\sum_{k,n\in\mathbf{N}}(c_{k,n,+}(f)\int_0^\xi\widehat{\varphi}_{k,n,+}(x)\,dx+c_{k,n,-}(f)\int_0^\xi\widehat{\varphi}_{k,n,-}(x)\,dx).$$

Using (1) i), (2), (4) and Fubini theorem, we get

$$\int_0^\xi\mathcal{F}f(x)\,dx=\sum_{k,n\in\mathbf{N}}(c_{k,n,+}(f)\int_{\mathbf{R}^+}\varphi_{k,n,+}(t)\,(\int_0^\xi\psi_x(t)\,dx)\,dt$$

$$+c_{k,n,-}(f)\int_{\mathbf{R}^+}\varphi_{k,n,-}(t)\,(\int_0^\xi\psi_x(t)\,dx)\,dt)$$

$$= \sum_{k,n \in N} (\langle f, \varphi_{k,n,+} \rangle \langle \varphi_{k,n,+}, D_\xi \rangle + \langle f, \varphi_{k,n,-} \rangle \langle \varphi_{k,n,-}, D_\xi \rangle) = \langle f, D_\xi \rangle,$$

where

$$D_\xi(t) := \int_0^\xi \psi_x(t)\, dx \quad (t \in \mathbf{R}^+)$$

is the Walsh-Dirichlet kernel (see e. g. [2]). The a. e. differentiability of the integral function implies the following

Theorem. *For every function $f \in L^2(\mathbf{R}^+)$ we have*

$$\mathscr{F} f(\xi) = \frac{d}{d\xi} \left(\int_{\mathbf{R}^+} f(t) D_\xi(t)\, dt \right) \quad (\text{a. e. } \xi \in \mathbf{R}^+).$$

This formula is an analogue of the following one for the classical Fourier transform of the function $f \in L^2(\mathbf{R})$:

$$\mathscr{F} f(\xi) = \frac{d}{d\xi} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(t) \frac{e^{-it\xi} - 1}{-it}\, dt \right) \quad (\text{a. e. } \xi \in \mathbf{R})$$

(see e. g. [5]).

REFERENCES

1. N. J. Fine. The generalized Walsh functions. *Trans. Amer. Math. Soc.*, **69**, 1950, 66-77.
2. J. Pál. On a concept of a derivative among functions defined on the dyadic field. *SIAM J. Math. Anal.*, **8**, 1977, 375-391.
3. J. Pál. The eigenfunctions of the Walsh-Fourier transform. — In: Approximation and Function Spaces. (Proc. of the Conf., Gdansk, 1979), 1981, 553-557.
4. J. Pál. On the a. e. convergence of the Fourier series with respect to the eigenfunctions of the Walsh-Fourier transform. — In: Functions, Series, Operators (Proc. of the Conf., Budapest, 1980) (to appear).
5. B. Szőkefalvi-Nagy. Introduction to Real Functions and Orthogonal Expansions. Oxford, 1965.

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