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AVERAGE MODULI AND THEIR FUNCTION SPACES

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Summary. An interpolation theorem for the average moduli is given. The notion 'one-sided K-functional' is introduced and is shown that in the simplest one-dimensional case it is equivalent to the average modulus of first order. There are introduced function spaces generated by means of the average moduli and the one-sided K-functional. The simplest connection between these spaces and Besov spaces is displayed.

1. Introduction. The average moduli (in one-dimensional case) are defined as follows:

Let f be a bounded function in the interval $[a, b]$. The k-th average modulus (or k-th τ-modulus) of f in L_p $(1 \leq p \leq \infty)$ is given by

$$\tau_k(f;\delta)_{L_p} = \|\omega_k(f, x;\delta)\|_{L_p[a,b]}, \tag{1}$$

where

$$\omega_k(f, x;\delta) = \sup\{|\Delta_h^k f(t)| : t, t+kh \in [x-k\delta/2,\ x+k\delta/2] \cap [a, b]\},$$

$$\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(t+mh).$$

For the history of the average moduli and their properties see [1-8]. In the last years was stated that these moduli have many applications in different domains, for example:

1. Approximation of functions by means of linear positive operators [1, 2].
2. Approximation of functions by means of piecewise monotone functions [3].
3. One-sided approximation of functions [4-8].
4. Numerical integration (quadrature formulae) [7-9].
5. Numerical solution of differential equations [10, 11].
6. Estimations for the error of approximation of summation linear operators, for example Bernstein operators, spline interpolation operators [12] (see also the paper of A. S. Andreev in this volume).

In this report we shall discuss the reasons, for which, in our opinion, the average moduli play (and will play) essential role in different domains.

2. Basic Interpolation Theorem. The following interpolation theorem is very essential for the average moduli:

Theorem 1. *Let L_n be a linear operator, which satisfies:*

a) *if f is a bounded integrable function in $[a, b]$, then $L_n(f; \cdot) \in L_p[a, b]$;*

b) $\| L_n(f; \cdot)\|_{L_p} \leq K \| f \|_{l^p_{\Sigma_n}}$, *where K is an absolute constant and*

$$\| f \|_{l^p_{\Sigma_n}} = \{ \sum_{i=1}^{n} | f(x_i) |^p \Delta_i \}^{1/p}, \ \Delta_i = x_{i+1} - x_i ;$$

$\Sigma_n = \{a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b\}$ *is a partition of the interval $[a, b]$;*

c) *if $f \in W^r_p$ ($f \in W^r_p$, if $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_p$), then*

$$\| L_n(f; \cdot) - f \|_{L_p} \leq K_r d^s_n \| f^{(r)} \|_{L_p}, \quad s \leq r,$$

where $d_n = \max_{1 \leq i \leq n} \Delta_i$ and K_r is an absolute constant, which depends only on r.

Then for every bounded integrable function f in $[a, b]$ the following estimation holds (if $d_n \leq 1$):

$$\| L_n[f; \cdot) - f \|_{L_p} \leq c \tau_r(f; d_n^{s/r})_{L_p}.$$

The constant c depends only on r, K, K_r.

Here and in what follows we shall separate the bounded function f from its class of equivalence in L_p and we assume that every function is given by its values in every point of its domain of definition.

Let us consider (using the same notations as in Theorem 1) the following K-functional (compare with Peetre [13, 14]):

$$K_{\Sigma_n}(f; t) = \inf_{f = f_1 + f_2} \{ \| f_1 \|_{l^p_{\Sigma_n}} + t \| f_2^{(r)} \|_{L_p} \}.$$

We have the following

Theorem 2. $K_{\Sigma_n}(f; d^r_n) \leq c(r) \tau_r(f; d_n)_{L_p}$, *where the constant $c(r)$ depends only on r.*

The proof of Theorem 2 is similar to the proof of Theorem 1, which is given in [12].

It is interesting, that if we consider the functional

$$K_{\mathscr{L}_t}(f; t^r) = \inf_{f = f_1 + f_2} \sup_{\Sigma_n \in \mathscr{L}_t} \{ \| f_1 \|_{l^p_{\Sigma_n}} + t^r \| f_2^{(r)} \|_{L_p} \}, \tag{2}$$

then this functional is equivalent to $\tau_r(f; t)_{L_p}$. Here $\mathscr{L}_t$ denotes the set of all partitions Σ_n of the interval $[a, b]$, for which $d_n \leq t$.

We see that (2) gives another definition of the τ-moduli as modifying K-functionals between the 'little' l^p spaces and the Sobolev spaces W^r_p.

3. One-Sided K-functional. The other reason, for which the τ-moduli are useful, is that the average moduli can be defined, except the definitions (1) and (2), in another way.

Let G be a given set and let X_1 and X_2 be two linear spaces of real-valued functions on G. We assume that X_1 and X_2 contain the constant functions. Let $\| \cdot \|_i$, $i = 1, 2$, are norms (or seminorms) in X_i, $i = 1, 2$.

We define the lower K-functional in X_1+X_2 for the norms (or seminorms) $\|\cdot\|_1$ and $\|\cdot\|_2$ by

$$K_+(f;t)=\inf_{f=f_1+f_2,\ f_1\geq 0}\{\|f_1\|_1+t\|f_2\|_2\}.$$

This lower K-functional is meaningful for all functions $f\in X_1+X_2$ for which $\inf\{f(x):x\in G\}>-\infty$. The upper K-functional in X_1+X_2 for the norms (or seminorms) $\|\cdot\|_1$ and $\|\cdot\|_2$ is given by

$$K_-(f;t)=\inf_{f=f_1+f_2,\ f_1\leq 0}\{\|f_1\|_1+t\|f_2\|_2\}.$$

The upper K-functional is meaningful for all functions $f\in X_1+X_2$, for which $\sup\{f(x):x\in G\}<\infty$. We define the one-sided K-functional in X_1+X_2 for the norms (or seminorms) $\|\cdot\|_1$ and $\|\cdot\|_2$ by $\tilde{K}(f;t)=\max\{K_+(f;t), K_-(f;t)\}$. This functional is meaningful for all bounded functions in X_1+X_2. The following theorem gives the most elementary connection between the average modulus $\tau_1(f;\delta)_{L_1}$ and the one-sided K-functional:

Theorem 3. *Let G be the interval* $[0, 1]$, $X_1=L[0, 1]$ *(see the remark after Theorem* 1), $X_2=W_1^1$, $\|f\|_1=\|f\|_{L_1}, \|f\|_2=\|f'\|_{L_1}$ *(seminorms in X_1, correspondingly X_2). Then for every bounded integrable function f on* $[0, 1]$ *we have*

$$c_2\tilde{K}(f;t)\leq\tau_1(f;t)_{L_1}\leq c_1\tilde{K}(f;t), \tag{3}$$

where c_1, c_2 are absolute constants (for example $c_1\leq 6$, $c_2\geq 1/12$).

Proof. Let f be a bounded integrable function on $[0, 1]$. We define the functions $S_n(f;x)$ and $J_n(f;x)$, as follows (compare with [6]):

Let us set (n — natural number): $x_i=i/n$, $i=0,\ldots,n$, $z_i=(x_{i-1}+x_i)/2$, $i=1,\ldots,n$, $y_i=\sup\{f(t):t\in[x_{i-1}, x_i]\}$, $\theta_i=\inf\{f(t):t\in[x_{i-1}, x_i]\}$. We set

$$S_n(f;x)=\begin{cases} y_1 & \text{for } x\in[0, z_1];\\ y_i & \text{for } x=z_i,\ i=1,\ldots,n;\\ y_n & \text{for } x\in[z_n, 1];\\ \max\{y_i, y_{i+1}\} & \text{for } x=x_i,\ i=1,\ldots,n-1;\\ \text{linear and continuous for } x\in[x_{i-1}, z_i], & \\ i=1,\ldots,n, \quad \text{and} \quad x\in[z_i, x_i],\ i=1,\ldots,n-1, & \end{cases}$$

$$J_n(f;x)=\begin{cases} \theta_1 & \text{for } x\in[0, z_1];\\ \theta_i & \text{for } x=z_i,\ i=1,\ldots,n;\\ \theta_n & \text{for } x\in[z_n, 1];\\ \min\{\theta_i, \theta_{i+1}\} & \text{for } x=x_i,\ i=1,\ldots,n-1;\\ \text{linear and continuous for } x\in[x_{i-1}, z_i], & \\ i=1,\ldots,n, \quad \text{and} \quad x\in[z_i, x_i],\ i=1,\ldots,n-1. & \end{cases}$$

We have obviously

$$J_n(f;x)\leq f(x)\leq S_n(f;x),\quad x\in[0, 1] \tag{4}$$

$J_n'(f;x)$ and $S_n'(f;x)$ exist for every $x\neq z_i$, $i=1,\ldots,n$, $x\neq x_i$, $i=1,\ldots, n-1$, and

$$|J_n'(f;x)|\leq 2n\omega(f,x;2/n);$$

(5) $$|S_n'(f;x)|\leq 2n\omega(f,x;2/n);$$

$$0\leq S_n(f;x)-J_n(f;x)\leq\omega(f,x;2/n).$$

From (4), (5) it follows $K_+(f;1/n)\leq\|f-J_n\|_{L_1}+\|J_n'\|_{L_1}/n\leq\tau_1(f;2/n)_L+2\tau_1(f;2/n)_L\leq 6\tau_1(f;1/n)_L$ (we use that $\tau_1(f;2\delta)_L\leq 2\tau_1(f;\delta)_L$, see [6]).

By analogical way $K_-(f;1/n)\leq 6\tau_1(f;1/n)_L$ and consequently $\tilde{K}(f;1/n)\leq 6\tau_1(f;1/n)_L$.

For arbitrary t we have $(1/(n+1)<t\leq 1/n)$:

$$\tilde{K}(f;t)\leq\tilde{K}(f;1/n)\leq 6\tau_1(f;1/n)_L=6\tau_1(f;\frac{1}{nt}t)_L\leq 12\tau_1(f;t)_L,$$

which proves the left side of (3).

To prove the right side of (3) we use that $\tau_1(f+g;\delta)_L\leq\tau_1(f;\delta)_L+\tau_1(g;\delta)_L$ and $\tau_1(f;\delta)_L\leq\delta\|f'\|_L$ (see [6]). Let $f_2^+\in W_1^1$ and $f_2^-\in W_1^1$ be such that $f_2^+(x)\leq f(x)\leq f_2^-(x)$, $x\in[0,1]$. We set $f_1^+=f-f_2^+$, $f_1^-=f-f_2^-$. We have

$$\tau_1(f;\delta)_L\leq\tau_1(f_1^+;\delta)_L+\tau_1(f_2^+;\delta)_L\leq\tau_1(f_1^+;\delta)_L+\delta\|(f_2^+)'\|_L,$$

$$\tau_1(f;\delta)_L\leq\tau_1(f_1^-;\delta)_L+\tau_1(f_2^-;\delta)_L\leq\tau_1(f_1^-;\delta)_L+\delta\|(f_2^-)'\|_L$$

or

(6) $$\tau_1(f;\delta)_L\leq\frac{1}{2}\{\tau_1(f_1^+;\delta)_L+\tau_1(f_1^-;\delta)_L+\delta(\|(f_2^+)'\|_L+\|(f_2^-)'\|_L\}.$$

Now we must estimate $\tau_1(f_1^+;\delta)_L$ and $\tau_1(f_1^-;\delta)_L$. We have

(7) $$\tau_1(f_1^+;\delta)_L+\tau_1(f_1^-;\delta)_L=\int_0^1\{\omega(f_1^+,x;\delta)+\omega(f_1^-,x;\delta)\}dx.$$

Let $x\in[0,1]$ and $y\in[0,1]$ be such that $|x-y|\leq\delta/2$. For $f_1^+(x)-f_1^+(y)$ we have two cases:

a) $0\leq f_1^+(x)-f_1^+(y)\leq f_1^+(x)$;

b) $0\leq f_1^+(y)-f_1^+(x)=f(y)-f_2^+(y)-f(x)+f_2^+(x)\leq f_2^-(y)\pm f_2^-(x)-f(x)+\omega(f_2^+,x;\delta)\leq f_2^-(x)-f(x)+\omega(f_2^-,x;\delta)+\omega(f_2^+,x;\delta)$

or in all two cases we have

$$|f_1^+(x)-f_1^+(y)|\leq|f_1^+(x)|+|f_1^-(x)|+\omega(f_2^-,x;\delta)+\omega(f_2^+,x;\delta).$$

From here it follows

(8) $$\omega(f_1^+,x;\delta)\leq 2\{|f_1^+(x)|+|f_1^-(x)|+\omega(f_2^-,x;\delta)+\omega(f_2^+,x;\delta)\}.$$

By analogical way we obtain

(9) $$\omega(f_1^-,x;\delta)\leq 2\{|f_1^+(x)|+|f_1^-(x)|+\omega(f_2^-,x;\delta)+\omega(f_2^+,x;\delta)\}.$$

From (6)-(9) we obtain

(10) $$\tau_1(f;\delta)_L\leq 2\{\|f_1^+\|_L+\|f_1^-\|_L\}+3\delta(\|(f_2^+)'\|_L+\|(f_2^-)'\|_L).$$

Since f_2^+ and f_2^- are arbitrary functions in W_1' such that $f_2^+ \leq f \leq f_2^-$, from (10) we obtain

$$\tau_1(f;\delta)_L \leq 3(K_+(f;\delta)+K_-(f;\delta)) \leq 6\tilde{K}(f;\delta).$$

Theorem 3 is proved.

4. Function Spaces, Connected with the Average Moduli and the One-Sided K-Functional. We introduce, by analogical way as intermediate spaces and Besov spaces, the corresponding function spaces using the average moduli and the one-sided K-functional.

The space $A_{\theta,p,q,r}$ is the space of all bounded functions on $[0, 1]$, for which

$$\|f\|_{\theta,p,q,r} = \{\int_0^\infty (t^{-\theta}\tau_r(f;t)_{L_p})^q \frac{dt}{t}\}^{1/q} < \infty$$

and the space $A_{\theta,q,\tilde{K}}$ for a given one-sided K-functional is the space of all bounded functions on the set G, for which

$$\|f\|_{\theta,q,\tilde{K}} = \{\int_0^\infty (t^{-\theta}\tilde{K}(f;t))^q \frac{dt}{t}\}^{1/q} < \infty.$$

Let us mention that it is possible to define τ-moduli also for the functions of n-variable and as consequence of the corresponding spaces $A_{\theta,p,q,r}$.

For one-dimensional case it is easy to obtain some imbedding theorems for the spaces $A_{\theta,p,q,r}$ and their connection with Besov spaces. We shall restrict ourselves only to the following

Theorem 4. *In the one-dimensional case the spaces $A_{\theta,p,q,r}$ for $\theta > 1$, $r > 1$ are equal (to equivalent norms) to the Besov spaces $B_{\theta,p,q,r}$.*

Let us remember that $f \in B_{\theta,p,q,r}$, if

$$|||f|||_{\theta,p,q,r} = \{\int_0^\infty (t^{-\theta}\omega_r(f;t)_{L_p})^q \frac{dt}{t}\}^{1/q} < \infty,$$

where

$$\omega_r(f;t)_{L_p} = \sup_{0<h\leq t} \{\int_0^{1-rh} |\Delta_h^r f(x)|^p dx\}^{1/p}.$$

The proof of Theorem 4 is based on the following inequalities: if $f' \in L_p$, then

(11) $$\omega_r(f;\delta)_{L_p} \leq \tau_r(f;\delta)_{L_p} \leq c\delta\omega_{r-1}(f';\delta)_{L_p},$$

where c is an absolute constant. For the proof of (11) in the case $r=2$ see [15].

We set the following problems:

1. To study the spaces $A_{\theta,p,q,r}$, $A_{\theta,q,\tilde{K}}$ in n-dimensional case and to find all imbedding theorems connected with them.
2. To find (in n-dimensional case) the connection between these spaces and Besov spaces.

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