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QUANTITATIVE THEOREMS OF KOROVKIN TYPE IN BOUNDED FUNCTION SPACES

M. A. Jiménez Pozo

Summary. This paper presents a collection of quantitative theorems of Korovkin type in spaces of bounded functions on a metric space. Estimates of the degree of convergence are obtained for sequences of operators in terms of the sup norm, the modulus of continuity and the coefficient of the convex deformation of the metric space. The asymptotic behaviour of the estimates are discussed and in some cases they are proved to be exact. These estimates improve some earlier results in the subject.

Introduction. As it is known, after the striking results of Korovkin [1] on the convergence of a sequence of positive linear operators in $C[a, b]$, a wide theory about the subject has been developed. Further, we will give some references, from which this development may be reconstructed.

One aspect of the theory is conformed by the quantitative results. They are estimates of the degree of convergence of the sequence of operators and often contain earlier results of qualitative type. These estimates begin with the paper of Mamedov [2] and continue with Freud [3] and with results of Shisha and Mond [4, 5] in $C[a, b]$ and $C_{2\pi}$, which have played an important role in the development of the estimates. Some horizontal extensions of the precedent results have been obtained by Ditzian [6], De Vore [7], Mohapatra [8] and others. Vertical generalizations are due to Censor [9], when $[a, b]$ is replaced by a compact convex subset of $\boldsymbol{R}^m$, by the author [10] for general metric spaces, by Nishishiraho [11] for a convex compact subset of a real locally convex Hausdorff space and others.

General Estimate of the Degree of Convergence. Qualitative theorems of Korovkin type in spaces of continuous functions have found a satisfactory form by means of different envelope techniques (cf. [12-14]). This is the technique we had in mind in the definition of test family in order to have quantitative results (cf. [10]).

In the sequel, let (X, d) be a metric space and Z, Y, nonempty subsets of X, such that $Z \subset Y$. Denote $B(X)$ the real space of bounded real functions on X.

Definition. Any family $\{f_z;\ z \in Z\} \subset B(X)$ is said to be a test family on Z, if $f_z(z) = 0$ for all $z \in Z$ and if it satisfies the following uniform condition: There exists a function $\varphi: R_+^* \to R_+^*$, such that $f_z(x) \geq \varphi(\alpha)$, whenever $x \in X$, $z \in Z$ and $d(x, z) \geq \alpha > 0$. Most important examples arise from

(1) $$f_z(x) = d^p(x, z), \quad \varphi(\alpha) = \alpha^p, \quad p \geq 1,$$

when X is bounded, and from

(2) $$f_z(x) = |\sin((x-z)/2)|^p, \quad \varphi(\alpha) = (\alpha/\pi)^p, \quad p \geq 1,$$

in the space of periodic functions, where

(3) $$X =]-\pi, \pi], \quad d(x, z) = \min\{|x-z|, 2\pi - |x-z|\}.$$

In the sequel, we will also consider a sequence of operators $L_n: E \to B(Y)$, where E is a linear subspace of $B(X)$, which contains a test family $\{f_z;\ z \in Z\}$ and the constant function 1. Each L_n is monotonous (i. e. $L_n f \geq L_n g$, whenever $f \geq g$) and its restriction to the linear hull of $\{f_z;\ z \in Z\}$ and 1 is linear. Finally, we denote $\| f \|_Z = \sup\{|f(z)|:\ z \in Z\}$. Under these assumptions let us consider simultaneously some cases: If $X = Y$ — the uniform convergence on Z. In particular, uniform convergence takes place, if $Z = X$ and if Z is a singleton — the pointwise convergence. If Y is a singleton, then convergence of functionals follows.

We shall use the modulus of continuity

(4) $$\omega(f, Z, \alpha) = \sup\{|f(x) - f(y)|;\ d(x, y) \leq \alpha, d(x, Z) \leq \alpha, d(y, Z) \leq \alpha\}$$

for each $f \in B(X)$ and $\alpha > 0$. Also we shall define $\omega(f, Z, 0) = \lim_{\alpha \to 0} \omega(f, Z, \alpha)$.

Theorem 1. *For every $f \in E$ and $\alpha > 0$ we have*

(5) $$\| L_n f - f \|_Z \leq \omega(f, Z, \alpha) \| L_n 1 \|_Z + \| f(L_n 1 - 1) \|_Z$$
$$+ 2 \| f \|_X (\varphi(\alpha))^{-1} \sup\{L_n(f_z, z);\ z \in Z\}.$$

Proof. Fix $z \in Z$ and $\delta = 2 \| f \|_X$. Define

$$R = \delta(\varphi(\alpha))^{-1} f_z + \omega(f, Z, \alpha), \quad g = f(z) + R, \quad h = f(z) - R.$$

It follows that $g, h \in E$ and $h \leq f \leq g$. Fix L_n. Since L_n is monotonous, then $L_n h \leq L_n f \leq L_n g$. Using now the linear property of L_n, we obtain an estimate of $|L_n(f, z) - f(z)|$ and then we take sup norm on Z.

Corollary 1. *If* $\| L_n 1 - 1 \|_Z \xrightarrow[n]{} 0$ and $L_n(f_z, z) \xrightarrow[n]{} 0$ *uniformly on* Z, *then* $\| L_n f - f \|_Z \xrightarrow[n]{} 0$ *for every* $f \in E$, *such that* $\omega(f, Z, 0) = 0$.

Corollary 2. (A version of the Localization Principle). *Let* $\varepsilon > 0$ *and* $Z(\varepsilon) = \{x;\ d(x, Z) \leq \varepsilon\}$. *Suppose the hypothesis of Corollary 1 holds and that each L_n is linear in the whole space E. Let $u, v \in E$ be such that $u = v$ on $Z(\varepsilon)$ for some $\varepsilon > 0$. Then* $\| L_n u - u \|_Z \xrightarrow[n]{} 0$ *if and only if* $\| L_n v - v \|_Z \xrightarrow[n]{} 0$.

Metrically Convex Spaces. In order to improve the estimate (5), one needs the inequality

(6) $$\omega(f, X, \lambda\alpha) \leq [\lambda + 1]\omega(f, X, \alpha);\ \lambda \geq 1, \alpha > 0,$$

where $[\cdot]$ denotes the integer part. This was unfortunately omitted in our first note [10].

Inequality (6) holds for every $f \in B(X)$, when X is a convex subset of $\boldsymbol{R}^m$. The key of the proof is that for any $x, y \in X$ and positive numbers a and b, such that $d(x, y) = a + b$, there exists $z \in X$, such that $d(x, z) = a$ and $d(z, y) = b$. Since this condition is independant of the algebraic structure of $\boldsymbol{R}^m$, it is taken as definition of convexity in general metric spaces. For example, the space considered in (3), is metrically convex. Of course, if (X, d) is metrically convex, then (6) holds for every $f \in B(X)$. Any convex subset of a normed space is metrically convex, but perhaps there may be others. However the mentioned condition is a characterization of convex sets in $\boldsymbol{R}^m$, then the above definition does not improve the situation in the subsets of $\boldsymbol{R}^m$. We have given the following generalization (cf. [15]).

Definition. Consider the natural generalization of a rectifiable arc Γ in (X, d) and its length $l(\Gamma)$. We shall say that (X, d) has a coefficient of convex deformation ρ and denote it by $D(X, d) = \rho$, if for every $x, y \in X$ there exists at least a rectifiable arc Γ_{xy} with extreme points x and y, and if

$$\sup_{x \neq y} \inf_{\Gamma_{xy}} l(\Gamma_{xy})/d(x, y) = \rho.$$

The best way to interpret geometrically this idea is to see that a semicircle in $\boldsymbol{R}^2$ has a coefficient $\pi/2$. It is proved in [15]:

Theorem 2. *If (X, d) is compact, then (X, d) is metrically convex if and only if $D(X, d) = 1$.*

On the other hand, a slight modification in the proof of Theorem 2 of [15] lets us rewrite it in a more general form:

Theorem 3. *If $D(X, d) = \rho$ or if $D(X', d) = \rho$ for some dense subspace (X', d) of the completion of (X, d), then*

$$\omega(f, X, \lambda\alpha) \leq [\rho\lambda + 1]\, \omega(f, X, \alpha); \ \lambda \geq 1, \ \alpha > 0 \tag{7}$$

holds for every $f \in B(X)$.

Combining the last two theorems, we obtain a generalization of (6). Although we may state complementary hypothesis for general test families in order to prove the following theorem, we will suppose for simplicity that $f_z(x) = d^p(x, z)$ and $\varphi(\alpha) = \alpha^p$, for some $p \geq 1$ fixed.

Theorem 4. *Suppose* (7) *holds in a bounded metric space (X, d). Then*

$$\| L_n f - f \|_Z \leq \omega(f, X, \alpha_n) \| L_n 1 + \rho A \|_Z + \| f(L_n 1 - 1) \|_Z \tag{8}$$

for every $f \in E$, where A is an arbitrary positive constant and

$$\alpha_n^p = A^{-1} \sup \{ L_n(d^p(\cdot, z), z); \ z \in Z \}. \tag{9}$$

Proof. It runs as in Theorem 1. Let $\alpha > 0$, $z \in Z$ and L_n be fixed. Construct g and h with $\delta = \rho\omega(f, X, \alpha)$. Now, it follows from (7) that $h \leq f \leq g$. By this way we will arrive at

$$\begin{aligned} \| L_n f - f \|_Z &\leq \omega(f, Z, \alpha) \| L_n 1 \|_Z + \| f(L_n 1 - 1) \|_Z \\ &+ \rho\omega(f, X, \alpha)\alpha^{-p} \sup \{ L_n(d^p(\cdot, z), z); \ z \in Z \}. \end{aligned} \tag{10}$$

If $\sup\{L_n(d^p(\cdot, z), z); \ z \in Z\} > 0$, we take $\alpha = \alpha_n$ as in (9), and (8) follows from (10). Otherwise, since (10) is true for every $\alpha > 0$, one only needs to take infimum in the right-hand side of (10)

We remark that the idea of a multiplicative constant A is due to Mond. This may help sometimes to improve the estimate (cf. [16]).

The Case of A-Distance. When Z is a singleton $\{z\}$, Theorems 1 and 4 give an estimate of $|L_n(f, z)-f(z)|$. Then, in that inequality we may take integrals, or sup-norm with a weight function and so on, in order to have estimates with different norms. However, starting from Korovkin's ideas in [17], Sendov [18] has given the following unifier definition.

Definition. Any relation r_A between the elements of a space S of real functions on (X, d) is said to be an A-distance in S on Z, if r_A is a pseudodistance which verifies the two conditions: (i) If for every $z\in Z$, $\varphi(z)\leq f(z)\leq \psi(z)$ and $\varphi(z)-C\leq g(z)\leq \psi(z)+C$, where C is a constant, then $r_A(f, g)\leq r_A(\varphi, \psi)+|C|$; (ii) If C is a constant, then $r_A(f, f+C)=0$ if and only if $C=0$. It follows that uniform distance, every L^p-distance ($p\geq 1$), uniform distance with a weight $\Theta(x)\geq 1$ and others, are A-distances. Then the main quantitative theorem of Sendov states that:

Theorem 5. *If L_n are positive linear operators on S and $f\in S$, then, for every $\alpha>0$*

$$r_A(L_n f-f)\leq \tau_A(f, 2\alpha)+\|f(L_n 1-1)\|_Z+\sup\{L_n(H_z, z);\ z\in Z\}, \tag{11}$$

where τ_A is a modulus of A-continuity, which coincides with $\omega(f, X, \alpha)$ in the case of the uniform distance and

$$H_z(x)=\begin{cases}\omega\ (f, X, d(x, z)-\alpha) & \text{for } d(x, z)>\alpha,\\ 0 & \text{for } d(x, z)\leq\alpha.\end{cases}$$

Sendov's estimate (11) is very sharp, but in applications it needs inequality (6) or similar ones because of the troublesome function H_z. If we suppose that (X, d) has a coefficient of convex deformation $\rho<\infty$, then it follows from (7) that $H_z(x)\leq\rho\omega(f, X, \alpha)\alpha^{-1}d(x, z)$. Now taking $\alpha=\alpha_n=\sup\{L_n(d(\cdot, z), z);\ z\in Z\}$, estimate (11) has the new form:

$$r_A(L_n f-f)\leq \tau_A(f, 2\alpha_n)+\|f(L_n 1-1)\|_Z+\rho\omega(f, X, \alpha_n). \tag{12}$$

In particular, when $r_A=\|\cdot\|_Z$, we arrive again to Theorem 4. Before concluding we will point out that to obtain estimates in general L^p-spaces, where elements are classes of integrable functions, we need another technique (cf. [19]).

Finite Systems of Test Functions. The striking aspect of Korovkin type theorems is that we may infer the strong convergence of a sequence of operators from the convergence on a finite test system. To obtain estimates in this case we have introduced a new metric (cf. [10]). We will restrict our attention to the main case in which X is a bounded subset of $\boldsymbol{R}^m$ and L_n are linear.

Let $G=\{f_1, \ldots, f_s\}$ be a finite set of continuous and separating real functions on the closure of X. Let

$$d_G^2(x, y)=\sum_{1\leq i\leq s}(f_i(x)-f_i(y))^2 \tag{13}$$

be defined on $X\times X$. Then, d_G is a distance on X, which defines the Euclidean topology. Now, we may reformulate Theorem 4 with respect to this new distance:

Theorem 6. *Suppose* (7) *holds in* (X, d_G). *Then*

(14) $$\| L_n f - f \|_Z \leq \omega(f, (X, d_G), \alpha_n) \| L_n 1 + \rho A \|_Z + \| f(L_n 1 - 1) \|_Z$$

for every $f \in E$ *and* $A > 0$, *where*

(15) $$\alpha_n^p = A^{-1} \sup \{ L_n(d_G^p(\cdot, z), z) ; z \in Z \}.$$

In particular, for $p = 2$, we have

(16) $$\alpha_n^2 A \leq \sum_{1 \leq i \leq s} (\| L_n f_i^2 - f_i^2 \|_Z + 2 \| f_i \|_Z \| L_n f_i - f_i \|_Z + \| f_i \|_Z \| L_n 1 - 1 \|_Z),$$

which implies that $1, f_i, f_i^2, 1 \leq i \leq s$, constitute a finite test system. Of course, we assume that E contains the test functions. Theorem 6 is a generalization of those ones of Censor, Shisha and Mond and others.

To apply Theorem 6, it will be convenient a criterion to be known, when (7) holds in (X, d_G). This is the meaning of the following result, whose proof is a straightforward task.

Theorem 7. *Let* $G(X) = \{(f_1(x), \ldots, f_s(x)) ; x \in X\} \subset \mathbf{R}^s$. *Then* $D(X, d_G) = \rho$ *if and only if* $D(G(X), d) = \rho$ *and* (X, d_G) *is metrically convex if and only if* $G(X)$ *is convex.*

We conclude with two remarks: (i) The inequality $\omega(f, (x, d_G), \alpha) \leq K\omega(f, (X, d), \alpha)$ for some absolutely constant K does not always hold. It depends on G. (ii) Estimates in the periodic case (or its m-dimensional version) with test functions 1, sin, cos, follow from Theorem 4 and the expressions $|x/\pi| \leq |\sin x/2| \leq |x/2|$, if $|x| \leq \pi$ and $\sin^2(x-z)/2 = (1 - \cos z \cos x - \sin x \sin z)/2$ or, also, from direct use of the test family (3).

On the Asymptotic Exactnees of the Estimates. Let (τ_n) and (τ_n') be two real sequences such that $\tau_n \underset{n}{\rightarrow} 0$ and $\tau_n' \underset{n}{\rightarrow} 0$. We will say that they are asymptotic equivalently and denote they by $\tau_n \sim \tau_n'$, if $\tau_n = O(\tau_n')$ and $\tau_n' = O(\tau_n)$. We search for the minimum permissive values of (α_n) and (δ_n) (save asymptotic equivalents) such that for every sequence L_n we have

(17) $$| L_n(f, z) - f(z) | = O(\omega(f, X, \alpha_n) + \delta_n).$$

From Theorem 4 it follows that

(18) $$\alpha_n = L_n(d(\cdot, z), z) \quad \delta_n = | L_n(1, z) - 1 |$$

are possible in (17) for $z \in Z$ fixed. Now, suppose (17) holds. Applying it to $f \equiv 1$, we obtain that $|L_n(1, z) - 1| = O(\delta_n)$. This means that $\delta_n = |L_n(1, z) - 1|$ are the minimum permissive values in (17) and that they are asymptotically exact for any sequence (L_n). Suppose again that (17) holds and that $\delta_n = 0$ for all n. Then $L_n(d(\cdot, z), z) = O(\omega(d(\cdot, z), X, \alpha_n)) = O(\alpha_n)$. This means that the values of α_n in (18) are the minimum permissive in (17), save asymptotic equivalents. Also it shows that they are asymptotically exact for any sequence (L_n) such that $L_n(1, z) \equiv 1$. But this exactness is not true in the general case as the following example shows. Take $X = [0, 1]$, $L_n f = f + n^{-1} f(1)$ and $z = 0$, then $|L_n(f, 0) - f(0)| = O(n^{-1})$. That is, we may take $\alpha_n \equiv 0$ in (17), but $L_n(d(\cdot, 0), 0) = n^{-1}$.

Now, let L_n be linear and $L_n(1, z) = 1$ for all n. Take $\beta_n = (L_n(d^2(\cdot, z), z))^{1/2}$, $\gamma_n = (L_n(d^4(\cdot, z), z))^{1/4}$ and α_n as in (18). Because of Theorem 4, these values of β_n and γ_n are permissive in (17) (in place of α_n). On the other hand, in many cases it is relatively easy to compute β_n and γ_n from a finite test system, as we saw in the last epigraph. But the Cauchy-Schwartz inequality for positive linear operators shows that $\alpha_n \leqq \beta_n \leqq \gamma_n$. Then it will be convenient to know a criterion for $\alpha_n \sim \beta_n$.

Theorem 8. *If* $\gamma_n = O(\beta_n)$, *then* $\beta_n = O(\alpha_n)$.

Proof. We may assume $\gamma_n \neq 0$ for all n. Let S be the linear hull of 1 and $d^k(\cdot, z)$, $k = 1, 2, 4$. Since $L_n(\cdot, z)$ are linear functionals on S, the interpolation formula [20] guarantees that there exist points $x_n^{(1)}, \dots, x_n^{(4)} \in X$ and nonnegative numbers $a_n^{(1)}, \dots, a_n^{(4)}$ such that $a_n^{(1)} + \cdots + a_n^{(4)} = 1$ and $L_n(f, z) = a_n^{(1)} f(x_n^{(1)}) + \cdots + a_n^{(4)} f(x_n^{(4)})$ for every $f \in S$. Take $u_n^{(j)} = d(x_n^{(j)}, z)$. We may suppose they are ordered in such a way that $(a_n^{(1)})^{1/2} u_n^{(1)} \geqq \cdots \geqq (a_n^{(4)})^{1/2} u_n^{(4)}$. Then $\beta_n \sim (a_n^{(1)})^{1/2} u_n^{(1)}$. If $v_n = \max\{(a_n^{(1)})^{1/4} u_n^{(1)}, \dots, (a_n^{(4)})^{1/4} u_n^{(4)}\}$, then $\gamma_n \sim v_n$. But for some $K > 0$ we have for every n that $(a_n^{(1)})^{1/4} u_n^{(1)} \leqq u_n \leqq K (a_n^{(1)})^{1/2} u_n^{(1)}$. This implies the existence of a constant $Q > 0$ such that $a_n^{(1)} \geqq Q$ for every n and then $\beta_n \sim u_n^{(1)}$. Since $a_n^{(1)} u_n^{(1)} = O(\alpha_n)$, also $u_n^{(1)} = O(\alpha_n)$ and the theorem follows. We remark that the above condition is not necessary. Take $[0, 1] = X$, $L_n f = a_n f + 2^{-1} f(n^{-3}) + n^{-4} f(n^{-1})$ where $a_n = 1 - 2^{-1} - n^{-4}$ for every $n > 1$. If $z = 0$, then $\alpha_n \sim \beta_n \sim n^{-3}$, but $\gamma_n \sim n^{-2}$.

Other estimates with sup-norm. Techniques that we have developed here may be succesfully carried out when (L_n) are linear, but not necessarily positive or when the space $B(X)$ is not necessarily real. Another important case arises, when we deal with strong convergence to operators different from the identity. For these cases see our papers [21] and [22].

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University of Havana
Department of Physics and Mathematics
Havana Cuba

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