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SEMICONTINUITY PROPERTIES OF THE STRONG BEST COAPPROXIMATION OPERATOR

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Summary. Let E be a normed linear space and G a linear subspace of E. This paper is concerned with a study of the semicontinuity properties of the set-valued strong best coapproximation operator. Some necessary and sufficient conditions in order that this operator is upper semicontinuous (lower semicontinuous) and upper (K)-semicontinuous (lower (K)-semicontinuous) have been provided. Certain conditions, which suffice to ensure that this operator is upper semicontinuous (lower semicontinuous), have been included. A related operator has been defined and its semicontinuity properties have been analysed.

1. Introduction. Let E be a normed linear space and G a linear subspace of E. The present paper is devoted to a study of the semicontinuity properties of the set-valued operator $R_{S,G}: x \to R_{S,G}(x)$, defined on the subset $\operatorname{dom} R_{S,G}=\{x\in E : R_{S,G}(x)\neq\emptyset\}$ of E, where $R_{S,G}(x)=\{g_0\in G: \|g_0-g\| + r\|x-g_0\|\leqq\|x-g\|$ for each $g\in G\}$ $(x\in\operatorname{dom} R_{S,G},\ 0<r\leqq 1)$. This study is analogous to the investigations made by Blatter et al. [1], Godini [6, 7], Morris [8], Deutsch et al. [2] and Singer [13, 14] in the case of the semicontinuity properties of the set-valued metric projection.

The mapping $R_{S,G}: x \to R_{S,G}(x)$ of $\operatorname{dom} R_{S,G}$ into the collection 2^G of all non-empty closed subsets of G is called the set-valued projection of dom $R_{S,G}$ onto G.

Recall that, if E and G are two metric spaces, then a mapping $\mathfrak{A}: E \to 2^G$ is called upper semicontinuous (u. s. c.), respectively, lower semicontinuous (l. s. c.), if the set $\{x\in E: \mathfrak{A}(x)\subset M\}$ is open for each open subset M of G, respectively, closed for each closed subset M of G, or, equivalently, if the set $\{x\in E:\mathfrak{A}(x)\cap N\neq\emptyset\}$ is closed for each closed subset N of G, respectively, open for each open subset N of G. $\mathfrak{A}$ is called upper (K)-semicontinuous (u. (K)-s. c.), respectively, lower (K)-semicontinuous (l. (K)-s. c.), if the relations $x_n\to x$, $g_n\in\mathfrak{A}(x_n)$ $(n=1, 2, \ldots)$, $g_n\to g_0$ imply $g_0\in\mathfrak{A}(x)$, respectively, if the relations $x_n\to x$, $g_0\in\mathfrak{A}(x)$ imply the existence of a sequence $\{g_n\}_1^\infty$ with $g_n\in\mathfrak{A}(x_n)$ $(n=1, 2, \ldots)$ such that $g_n\to g_0$.

If A is a subset of E, we denote by $\mathfrak{A}(A)=U\{\mathfrak{A}(x): x\in A\}$.

2. Upper Semicontinuity of $R_{S,G}$. In this section some conditions, under which the operator $R_{S,G}$ is u. s. c., are given.

Proposition 2.1. Let G be a closed linear subspace of a normed linear space E. Then $R_{S,G}$ is u. s. c., if any one of the following conditions holds:

(1) $\dim G<\infty$;

(2) $\dim E/G<\infty$;

(3) G is boundedly compact;

(4) $R_{S,G}^{-1}(0)=\{x\in \operatorname{dom} R_{S,G}: 0\in R_{S,G}(x)\}$ is boundedly compact.

Proposition 2.2. Let G be a closed linear subspace of a normed linear space E such that $R_{S,G}^{-1}(0)$ is boundedly compact. Then

(1) $R_{S,G}$ is u. s. c.;

(2) For each $x\in \operatorname{dom} R_{S,G}$, the set $R_{S,G}(x)$ is compact.

Remark. If $\dim G<\infty$ and $R_{S,G}$ is u. s. c., then G is boundedly compact.

Next some results, concerning the characterization of upper semicontinuity of the operator $R_{S,G}$, are provided.

Proposition 2.3. Let G be a closed linear subspace of a normed linear space E of finite codimension. Then $R_{S,G}$ is u. s. c. if and only if $R_{S,G}^{-1}(0)$ is boundedly compact.

Proposition 2.4. Let G be a closed linear subspace of a normed linear space E. Then $R_{S,G}$ is u. s. c. if and only if for each closed subset N of G, $N+R_{S,G}^{-1}(0)$ is closed.

The following theorem is analogous to Theorem 1 [14].

Theorem 2.1. *Let G be a closed linear subspace of a normed linear space E such that $R_{S,G}$ is u. s. c. Then for each $x\in \operatorname{dom} R_{S,G}$, the set $R_{S,G}(x)$ is compact.*

Next, it is shown that the converse of Theorem 2.1 holds, when G is a hyperplane.

Theorem 2.2. *Let G be a hyperplane in a normed linear space E such that for each $x\in \operatorname{dom} R_{S,G}$ the set $R_{S,G}(x)$ is compact. Then $R_{S,G}$ is u. s. c.*

The following theorem is an analogue to Theorem 1 [6].

Theorem 2.3. *Let G be a closed linear subspace of a normed linear space E. Then the following assertions are equivalent:*

(1) *$R_{S,G}$ is u. s. c.;*

(2) (a) *For each compact subset A of* $\operatorname{dom} R_{S,G}$, *the subset $R_{S,G}(A)=U\{R_{S,G}(x): x\in A\}$ is compact in G;*

(b) *The relations $x_n\to x$, $g_0\in R_{S,G}(x_n)$ $(n=1, 2, \dots)$ imply $g_0\in R_{S,G}(x)$.*

We now provide one result, pertaining to the upper (K)-semicontinuity of $R_{S,G}$.

Proposition 2.5. Let G be a closed linear subspace of a normed linear space E. Then

(1) $R_{S,G}$ is u. (K)-s. c.;

(2) For each compact subset A of $\operatorname{dom} R_{S,G}$, the subset $R_{S,G}(A)$ is closed in G.

Definition. Let G be a closed linear subspace of a normed linear space E. A set-valued mapping $\mathscr{V}_{S,G}$ of $\operatorname{dom} R_{S,G/G}$ into $2^{\overset{-1}{R}_{S,G}(0)}$ is defined by $\mathscr{V}_{S,G}(x+G)=x-R_{S,G}(x)=\{x-g_0: g_0\in R_{S,G}(x)\}$ $(x+G\in \operatorname{dom} R_{S,G/G})$.

The mapping $\mathscr{V}_{S,G}: \operatorname{dom} R_{S,G/G}\to 2^{\overset{-1}{R}_{S,G}(0)}$ is well defined, as $R_{S,G}(x+g)=R_{S,G}(x)+g$ $(g\in G)$. It can be easily seen that $\mathscr{V}_{S,G}(x+g)\subset 2^{\overset{-1}{R}_{S,G}(0)}$, i.e. $\mathscr{V}_{S,G}(x+G)$ is a non-empty closed subset of $R_{S,G}^{-1}(0)$. Since $R_{S,G}(x)\neq\emptyset$ for every

$x \in \operatorname{dom} R_{S,G}$, $\mathcal{V}_{S,G}(x+G) \neq \emptyset$. Further $\mathcal{V}_{S,G}(x+G) \subset R_{S,G}^{-1}(0)$, since for every $g_0 \in R_{S,G}(x)$, $0 \in R_{S,G}(x-g_0)$ and hence $x-g_0 \in R_{S,G}^{-1}(0)$.

Now it will be proved that $\mathcal{V}_{S,G}(x+G)$ is closed. Let $y_n \in \mathcal{V}_{S,G}(x+G)$ $(n=1, 2, \ldots)$ and $y_n \to y$. Then there exists $g_n \in R_{S,G}(x)$ $(n=1, 2, \ldots)$ such that $y_n = x - g_n$. Since $y_n \to y$, $\{g_n\}$ is bounded and $R_{S,G}(x)$ is closed, $g_n \to g' \in R_{S,G}(x)$, which shows that $x-g' \in \mathcal{V}_{S,G}(x+G)$. Thus $\mathcal{V}_{S,G}(x+G)$ is closed.

Statements, analogous to Proposition 2.1 and Theorem 2.1, can be established in the case of $\mathcal{V}_{S,G}$.

The following results deal with necessary and sufficient conditions in order that $\mathcal{V}_{S,G}$ is u. s. c.

Proposition 2.6. Let G be a closed linear subspace of a normed linear space E. Then $\mathcal{V}_{S,G}$ is u. s. c. if and only if for each compact subset $A \subset \operatorname{dom} R_{S,G/G}$ the subset $\mathcal{V}_{S,G}(A)$ is a compact subset of $\operatorname{dom} R_{S,G/G}$.

Proposition 2.7. Let G be a closed linear subspace of a normed linear space E. Then $\mathcal{V}_{S,G}$ is u. s. c. if and only if for each closed subset C of $R_{S,G}^{-1}(0)$, $C+G$ is closed.

Theorem 2.4. *Let G be a closed linear subspace of a normed linear space E. Then $R_{S,G}$ is u. s. c. if and only if $\mathcal{V}_{S,G}$ is u. s. c.*

3. Lower Semicontinuity of $R_{S,G}$. Obviously, an analogue to Proposition 2.1 can be proved in this context. The following lemma is an effective tool in proving the subsequent theorem.

Lemma 3.1. *Let G be a closed linear subspace of a normed linear space E. Then $R_{S,G}$ is l. s. c., if it is l. (K)-s. c.*

The next theorem spells out a condition under which the operator $R_{S,G}$ is l. s. c.

Theorem 3.1. *Let G be a hyperplane in a normed linear space E. Then $R_{S,G}$ is l. s. c.*

The following theorem is analogous to Theorem 2.4.

Theorem 3.2. *Let G be a closed linear subspace of a normed linear space E. Then $R_{S,G}$ is l. s. c. if and only if $\mathcal{V}_{S,G}$ is l. s. c.*

Next, we shall give some equivalent conditions for the lower semicontinuity of $R_{S,G}$ to exist.

Theorem 3.3. *Let G be a closed linear subspace of a normed linear space E. Then the following assertions are equivalent:*

(1) *$R_{S,G}$ is l. s. c.;*

(2) *For each open subset D of G, $D+R_{S,G}^{-1}(0)$ is an open subset of* $\operatorname{dom} R_{S,G}$;

(3) *For each open subset C of $R_{S,G}^{-1}(0)$, $C+G$ is an open subset of* $\operatorname{dom} R_{S,G}$;

(4) *For each subset N of G*

$$\overline{\{x \in \operatorname{dom} R_{S,G} : R_{S,G}(x) \subset N\}} \subset \{x \in \operatorname{dom} R_{S,G} : R_{S,G}(x) \subset \bar{N}\}.$$

(5) *For each relatively compact subset A of* $\operatorname{dom} R_{S,G}$

$$R_{S,G}(\bar{A}) \subset \overline{R_{S,G}(A)}.$$

(6) *For each subset A of* $\operatorname{dom} R_{S,G}$

$$R_{S,G}(\bar{A}) \subset \overline{R_{S,G}(A)}.$$

With regard to the lower (K)-semicontinuity the following is valid.

Proposition 3.2. Let G be a closed linear subspace of a normed linear space E. Then $R_{S,G}$ is l.(K)-s. c. if and only if $\mathcal{V}_{S,G}$ is l.(K)-s. c.

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