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ON SPACES OF FUNCTIONS AND DISTRIBUTIONS WITH DOMINATING MIXED SMOOTHNESS PROPERTIES OF BESOV-TRIEBEL-LIZORKIN TYPE

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Summary. The paper deals with function spaces, which appear in the theory of integral operators to describe the smoothness of the related kernel such that result mappings between Besov or Triebel-Lizorkin spaces. A survey on the main properties such as characterization of Fourier multipliers and imbeddings, which can be derived by means of Fourier analysis, is given. We have relations to mixed L_p-spaces and spaces of functions with a dominating mixed derivative.

1. Introduction and Definitions. In this paper we want to present some new results concerning spaces of functions and distributions with dominating mixed smoothness properties. The definition of spaces of such a type goes back to Nikolskij [7], Amanov [1] and Lizorkin, Nikolskij [6]. A comprehensive treatment can be found in Amanov [2]. In latter time the theory was forced by the papers of Triebel [14] (see also [15, Chapter 2), where a general approach, basing on methods of Fourier analysis, is given. A new point of view came in in connection with integral operators $(\mathfrak{K}f)(y) = \int_{R_n} K(y, x) f(y) dy$.

The question was to determine smoothness properties of the kernel $K(y, x)$, $y \in R_n$, $x \in R_m$, such that the related operator $\mathfrak{K}$ maps between Besov spaces. We refer to Pietsch [8]. Here m, n are natural numbers. R_n denotes the Euclidean n-space. Suitable spaces were found by Triebel [18]. But before we turn to the definition of these spaces let us agree some notations. If n is a natural number, we denote by Φ_n the class of all systems of infinitely differentiable test functions, satisfying the properties:

(i) $\operatorname{supp} \psi_0(y) \subset \{y \in R_n \mid |y| \leq 2\}$;

$\operatorname{supp} \psi_l(y) \subset \{y \in R_n \mid 2^{l-1} \leq |y| \leq 2^{l+1}\}$, $l = 1, 2, \ldots$,

(ii) $\sup_y |D^\alpha \psi_l(y)| \leq c_\alpha 2^{-l|\alpha|}$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, $|\alpha| = \Sigma_{j=1}^n \alpha_j$,

(iii) $\sum_{l=0}^{\infty} \psi_l(y) = 1$.

As usually $S(R_n)$ and $S'(R_n)$ denote the spaces of all rapidly decreasing functions and their topological dual the space of all tempered distributions, respectively. In the following by F and F^{-1} we shall denote the Fourier transform in $S'(R_{n+m})$ and its inverse, respectively. Let $0<p, q, u, v\leq\infty$, $-\infty<s, t<\infty$. Then we want to investigate the spaces

(1) $$B^s_{p,q}(R_m, B^t_{u,v}(R_n))=\{f\in S'(R_{n+m})\mid \exists\psi\in\Phi_n,\ \varphi\in\Phi_m \text{ such that}$$

$$\| f\mid B^s_{p,q}(B^t_{u,v})\|^{(\psi,\varphi)}=\| 2^{sk+tl}(F^{-1}\varphi_k(\xi)\psi_l(\eta)Ff)(y,x)\mid l_v(L_u)\mid l_q(L_p)\|<\infty\}$$

and if additionally $p, u<\infty$

(2) $$F^s_{p,q}(R_m, F^t_{u,v}(R_n))=\{f\in S'(R_{n+m})\mid \exists\psi\in\Phi_n,\ \varphi\in\Phi_m \text{ such that}$$

$$\| f\mid F^s_{p,q}(F^t_{u,v})\|^{(\psi,\varphi)}=\|2^{sk+tl}(F^{-1}\varphi_k(\xi)\psi_l(\eta)Ff)(y, x)\mid L_u(l_v)\mid L_p(l_q)\|<\infty\}.$$

Here $l_v(L_u)$ and $L_u(l_v)$ mean the L_u-valued l_v-space and the l_v-valued L_u-space. At first we have to take the quasinorms in $l_v(L_u)$ (or $L_u(l_v)$) with respect to integration over $y\in R_n$ and summation over $l=0, 1$ $2,\ldots$ and afterwards the quasinorms in $l_q(L_p)$ (or $L_p(l_q)$) with respect to integration over $x\in R_m$ and summation over $k=0, 1, 2,\ldots$.

Triebel [18] showed that

$$\mathfrak{K}:\quad B^{-t}_{u',v'}(R_n)\longrightarrow B^s_{p,q}(R_m)$$

(linear bounded mapping), if $K\in B^s_{p,q}(R_m, B^t_{u,v}(R_n))$, $1<u, v<\infty$, $1/u+1/u'=1$, $1/v+1/v'=1$. Here $B^s_{p,q}(R_m)$ denotes the isotropic Besov space. An analogous result is valid in the case of the F-spaces. Then we have a mapping property between the Triebel-Lizorkin spaces. See [9]. For the definition of the Besov and Triebel-Lizorkin spaces we refer to [17]. The problem, which appeared, was to give a detailed study of the spaces, defined in (1) and (2) in the spirit of the modern theory of function spaces (cf. [16, 17)], or, in other words, a theory of these spaces from the standpoint of Fourier analysis. This was done in the papers [9 — 12] and [14] of the author in a somewhat more general case (There functions of n variables were considered.). The aim of the following sections is to give a survey on the main results applied to the spaces of (1) and (2).

2. Basic Properties of $B^s_{p,q}(R_m, B^t_{u,v}(R_n))$ and $F^s_{p,q}(R_m, F^t_{u,v}(R_n))$. If we look at the definitions (1) and (2) it becomes clear that we have to deal with systems $\{f_{l,k}(y, x)\}$ of certain entire analytic functions of exponential type, belonging to the space $L_p(R_m, L_u(R_n))$, consisting of all measurable functions defined on R_{n+m}, such that

(3) $$\| g(y, x)\mid L_p(L_u)\|=\Big(\int_{R_m}\Big(\int_{R_n}|g(y, x)|^u\,dy\Big)^{p/u}dx\Big)^{1/p}<\infty,$$

$0<p, u\leq\infty$ (modification if $u=\infty$ or $p=\infty$). This is a space with mixed quasinorm in the sense of Benedek, Panzone [4]. Moreover, we have

$$\operatorname{supp} Ff_{l,k}\subset\{y\mid 2^{l-1}\leq|y|\leq 2^{l+1}\}\times\{x\mid 2^{k-1}\leq|x|\leq 2^{k+1}\}$$

(modification if $k=0$ or $l=0$) and

$$\| f_{l,k}(y, x)\mid l_v(L_u)\mid l_q(L_p)\|<\infty$$

in the case of the B-spaces or

$$\| f_{l,k}(y, x) \mid L_u(l_v) \mid L_p(l_q) \| < \infty$$

in the case of the F-spaces. The basic tools in the theory are Fourier multiplier theorems for spaces of systems with these properties, analogous to those ones for $L_p(l_q)$ in the theory of Triebel-Lizorkin spaces in [17]. They can be derived, using Bagby's extended maximal inequality and the method of Triebel [17, Proposition 2.2.3]. For details we refer to [11, 14]. As a consequence one obtains the following theorems.

Theorem 1 ([9, 14]). (i) *If f belongs to $B^s_{p,q}(R_m, (B^t_{u,v}(R_n))$ (or $F^s_{p,q}(R_m, F^t_{u,v}(R_n)))$, then it holds $\| f \mid \ldots \|^{(\psi,\varphi)} < \infty$ for all $\psi \in \Phi_n$ and $\varphi \in \Phi_m$. All quasinorms $\| f \mid \ldots \|^{(\psi,\varphi)}$ (norms if $1 \leq p, q, u, v \leq \infty$) are equivalent to each other on the corresponding spaces. Equipped with the quasinorms $\| f \mid \ldots \|^{(\psi,\varphi)}$ all spaces are quasi-Banach spaces (Banach spaces, if $1 \leq p, q, u, v \leq \infty$).*

(ii) *We have the topological imbeddings $S(R_{n+m}) \subset B^s_{p,q}(R_m, B^t_{u,v}(R_n))$, $F^s_{p,q}(R_m, F^t_{u,v}(R_n)) \subset S'(R_{n+m})$.* If *$p, q, u, v < \infty$, then $S(R_{n+m})$ is dense in both spaces.*

It is shown in [11] and [14] that we can weaken the assumptions concerning the systems $\{\psi_l(y)\}_l$ and $\{\varphi_k(x)\}_k$. Furthermore in the case of the B-spaces the complicated technique of maximal functions can be avoided. See [10, 14]. In particular this implies the following criterion on Fourier multipliers.

Theorem 2 ([10, 14]). *Let $0 < p, q, u, v \leq \infty$ ($p, u < \infty$ in the case of F-spaces) and $-\infty < s, t < \infty$. We put*

$$(4) \qquad \varkappa^B_n = n\left(\frac{1}{u} + \frac{1}{2}\right), \qquad \varkappa^B_m = m\left(\frac{1}{\min(u, v, p)} + \frac{1}{2}\right);$$

$$(5) \qquad \varkappa^F_n = n\left(\frac{1}{\min(u, v)} + \frac{1}{2}\right), \qquad \varkappa^F_m = m\left(\frac{1}{\min(u, v, p, q)} + \frac{1}{2}\right);$$

$$(6) \qquad \varkappa^*_n = n\left(\frac{1}{\min(u, 1)} - \frac{1}{2}\right), \qquad \varkappa^*_m = m\left(\frac{1}{\min(u, v, p, 1)} - \frac{1}{2}\right).$$

Further let $\mu(y, x)$ be an infinitely differentiable function on R_{n+m} and let L_n, L_m be natural numbers. If $L_n > \varkappa^B_n$ and $L_m > \varkappa^B_m$, then there exists a positive constant c such that

$$(7) \qquad \| F^{-1}(\mu(\eta, \xi) Ff) \mid B^s_{p,q}(B^t_{u,v}) \| \leq c M_{L_n, L_m} \| f \mid B^s_{p,q}(B^t_{u,v}) \|$$

for all $f \in B^s_{p,q}(R_m, B^t_{u,v}(R_n))$, where

$$M_{L_n, L_m} = \sup_{|\beta| \leq L_n,\ |\alpha| \leq L_m} \ \sup_{(y,x) \in R_{n+m}} (1 + |y|^2)^{|\beta|/2} (1 + |x|^2)^{|\alpha|/2} | D^{(\beta,\alpha)} \mu(y, x) |.$$

An analogous result is true for the F-spaces, if we replace $\varkappa^B_n$ by $\varkappa^F_n$ and $\varkappa^B_m$ by $\varkappa^F_m$. If we consider functions $\mu(y, x) = \mu_1(y)\mu_2(x)$, (7) is valid with $\varkappa^*_n$ and $\varkappa^*_m$ instead of $\varkappa^B_n$ and $\varkappa^B_m$, respectively.

3. Spaces with a Dominating Mixed Derivative. Next we want to make clear the relations to the spaces of functions with dominating mixed derivatives, mentioned at the beginning of Section 1.

Theorem 3 ([11, 14]. (i) *Let* $1<p,\ u<\infty$. *Then it holds*

$$F^0_{p,\min(u,2)}(R_m,\ F^0_{u,2}(R_n))\subset L_p(R_m,\ L_u(R_n))\subset F^0_{p,\ \max(u,2)}(R_m, F^0_{u,2}(R_n)).$$

(ii) *Let* $1<p,\ u<\infty,\ -\infty<t,\ s<\infty$. *Then we have*

$$F^s_{p,\ \min(u,2)}(R_m,\ F^t_{u,2}(R_n))\subset H^s_p(R_m, H^t_u(R_n))\subset F^s_{p,\ \max(u,2)}(R_m, F^t_{u,2}(R_n)),$$

where.

$$H^s_p(R_m, H^t_u(R_n))=\{f\in S'(R_{n+m})\,|F^{-1}[(1+|\eta|^2)^{t/2}(1+|\xi|^2)^{s/2}Ff]\in L_p(R_m, L_u(R_n))\}.$$

The theorem shows the relations to the well-known Sobolev-Lebesgue spaces with a dominating mixed derivative [6] as well as to the L_p-spaces with a mixed norm [4]. Note also that $B^s_{p,p}(R_1, B^t_{p,p}(R_1))$, where $1\leq p\leq\infty$, $0\leq s,\ t<\infty$, coincide with the spaces $S^{(t,s)}_{p,p}B(R_2)$ of Nikolskij [7] ($p=\infty$) and Amanov [1, 2] (cf. [10]). Moreover, we have representations of Nikoskij type, if $0<p,\ q,\ u,\ v\leq\infty$,

$$t>n\left(\frac{1}{\min(u,\ 1)}-1\right),\ s>m\left(\frac{1}{\min(u,\ v,\ p,\ 1)}-1\right)$$

for the B-spaces and in the case of F-spaces, if $0<p,\ u<\infty,\ 0<q,\ v\leq\infty$ and

$$t>n\,\frac{1}{\min(u,v)},\qquad s>m\,\frac{1}{\min(u,\ v,\ p,\ q)}.$$

4. Imbeddings and Traces. On the basis of inequalities of Nikolskij type for the systems $\{f_{l,k}(y,\ x)\}$, described in Section 2, it is possible to establish imbedding theorems for different metrics and different dimensions.

Theorem 4 ([12, 14]). *Let* $0<p_0\leq p_1\leq\infty,\ 0<u_0\leq u_1\leq\infty,\ 0<v,\ q\leq\infty$, $-\infty<s_1\leq s_0<\infty,\ -\infty<t_1\leq t_0<\infty$. *Further we suppose* $s_0-m/p_0=s_1-m/p_1$, $t_0-n/u_0=t_1-n/u_1$ *Then it holds*

$$B^{s_0}_{p_0,q}(R_m,\ B^{t_0}_{u_0,v}(R_n))\subset B^{s_1}_{p_1,q}(R_m,\ B^{t_1}_{u_1,v}(R_n)).$$

If additionally $p_1,\ u_1<\infty$ and $0<q_0,\ v_0,\ q_1,\ v_1\leq\infty$, then it holds $F^{s_0}_{p_0,q_0}(R_m, F^{t_0}_{u_0,v_0}(R_n))\subset F^{s_1}_{p_1,q_1}(R_m,\ F^{t_1}_{u_1,v_1}(R_n))$.

Combining Theorem 3 and Theorem 4, we get imbeddings into mixed L_p-spaces and Sobolev-Lebesgue spaces with a dominating mixed derivative.

Next we define the operators $\Re_m f=f(y,\ 0)$, $\Re_n f=f(0,\ x)$, if f belongs to $S'(R_{n+m})$ such that supp Ff is compact. Note that we can find dense subsets of such functions in our spaces.

Theorem 5 ([12, 14]). *Let* $0<p,\ u,\ q,\ v\leq\infty,\ -\infty<s,\ t<\infty$.

(i) *If* $s>m/p$, *then (after extension)* $\Re_m$ *is a retraction from* $B^s_{p,q}(R_m, B^t_{u,v}(R_n))$ *onto* $B^t_{u,v}(R_n)$ *and from* $F^s_{p,q}(R_m,\ F^t_{u,v}(R_n))$ *onto* $F^t_{u,v}(R_n)$.

(ii) *If* $t>n/u$, *then (after extension)* $\Re_n$ *is a retraction from* $B^s_{p,q}(R_m, B^t_{u,v}(R_n))$ *onto* $B^s_{p,q}(R_m)$ *and from* $F^s_{p,q}(R_m,\ F^t_{u,v}(R_n))$ *onto* $F^s_{p,q}(R_m)$.

Here $B^s_{p,q}(R_m)$, $F^s_{p,q}(R_m)$ denote the isotropic Besov and Triebel-Lizorkin spaces of [17].

5. Generalizations. The definitions (1) and (2) can obviously be generalized to functions of more than two variables. All results can be carried over without serious difficulties (see [10, 11, 12, 14]).

We have another possibilities to describe dominating mixed smoothness properties. For example, we can change the order of taking the quasinorms in l_q, l_v, L_p, and L_u in (1) or (2). Taking first the quasinorm $\| \cdot | L_p(L_u) \|$ and afterwards the quasinorm $\| \cdot | l_q(l_v) \|$, we obtain the generalization of the spaces of Nikolskij [7] and Amanov [1, 2] to mixed quasinorms and to values p, u less than one. These spaces are investigated in [13]. For a recent work we refer also to the paper of Fernandez [5], where a similar approach is given in the case $1 \leq p,\ u \leq \infty$.

On the other hand, we could take at first the quasinorm $\| \cdot | l_q(l_v) \|$ and then $\| \cdot | L_p(L_u) \|$. The resulting space generalize the spaces $H^s_p(R_m, H^t_u(R_n))$ from Theorem 3 [11, 14].

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