КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 513—518

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 513—518

CONVERGENCE OF NUMERICAL METHODS

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Let us have a situation, in which we are looking for an unknown function f from a metric space (X, r), where X is a set of functions, defined on the compact Ω, and r is a distance in X. If we use a numerical method for finding the function f, then the result will be usually a set of approximate values $\{y_i\}_1^m$ of f for a finite number of values $\{x_i\}_1^m$ of the argument x of f. More precisely, for every cardinal m and for every $\varepsilon>0$, for every choice of the points $\{x_i\}_1^m \subset \Omega$, the numerical method will produce a set of m numbers $\{y_i\}_1^m$, for which the inequalities $|f(x_i)-y_i|<\varepsilon$ are satisfied, $i=1, 2, 3, \ldots, m$.

If f^* is a function from X, for which the equalities $f^*(x_i)=y_i$; $i=1, 2, 3, \ldots, m$, are satisfied, then f^* can be considered as an approximation of the unknown function f. But as f and f^* are elements of a metric space, then the error of the numerical method we use should be evaluated by $r(f, f^*)$. That's why it is important to know at which conditions $r(f, f^*)$ can be made small enough at a convenient choice of ε and the points $\{x_i\}_1^m$. The purpose of this paper is to consider this problem in rather general setting.

1. Discrete Convergence and Metric Convergence. To describe the real situation, which arises in solving problems by numerical methods, involving unknown functions, we introduce the concept of discrete convergence of a sequence of functions $\{f_n\}_1^\infty$, defined in a compact Ω. We shall suppose, that in the compact Ω a distance $\rho(x, t)$, $x, t \in \Omega$ is defined. Then, for every $\varepsilon>0$ in Ω there is a finite ε-net*.

Definition 1. We say that the sequence $\{f_n\}_1^\infty$ of functions, defined in the compact Ω, is discretely convergent to the function f, if for every $\varepsilon>0$ there exist a finite ε-net $\Omega_m=\{x_i\}_1^m$ in Ω and a cardinal N, such that for every $n \geqq N$ and for every $x_i \in \Omega_m$ the inequality $|f_n(x_i)-f(x_i)|<\varepsilon$ is satisfied.

It is easy to see that in the general case, when we use a numerical method for finding an unknown function, in fact we find the elements of a sequence of

*A subset Ω' of Ω is named ε-net if for every $x \in \Omega$ there is $x' \in \Omega'$ such that $\rho(x, x') \leqq \varepsilon$.

functions, which is discretely convergent to the wanted function. But the wanted function is usually an element of a metric space. That's why it is important to know the conditions at which from a discrete convergence follows a convergence in respect to the distance in the metric space.

We shall consider this problem for metric spaces of functions, in which the distance is in agreement with the natural ordering of the functions. Such distances are considered by Korovkin [1] and are called A-distances. We'll use a slight modification of the concept A-distance [2].

Let X be a set of functions, defined in the compact Ω. The set X will be considered partially ordered by the natural order. If for every $x \in \Omega$ the inequality $f(x) \leq g(x)$ is satisfied, we shall write $f \leq g$.

Definition 2. We shall say that $r(f, g)$, $f, g \in X$, is an A-distance in X, if it satisfies the following conditions:

1) $r(f, g) = r(g, f)$;

2) $r(f, g) \leq r(f, h) + r(h, g)$, $f, g, h \in X$;

3) If $\varphi, \psi, f, g \in X$, $C = \text{const}$ and

$$\varphi \leq f \leq \psi,\ \varphi - C \leq g \leq \psi + C, \text{ then } r(f, g) \leq r(\varphi, \psi) + |C|;$$

4) If $C = \text{const}$, then

$$r(f, f+C) = 0 \Leftrightarrow C = 0.$$

In this definition conditions 1) and 2) are the usual conditions for a symmetry of the distance and the triangle inequality. It is typical, for an A-distance, the condition 3), which expresses the agreement between the distance and the order. The condition 4) shows when from $r(f, g) = 0$ follows, that f and g are considered non-different. The A-distance allows the equality $f(x) = g(x)$ to fail for some $x \in \Omega$ and nevertheless to have $r(f, g) = 0$, i. e. f and g not to coincide, but not to be distinguished between each other in respect to this distance.

It is easy to check that the classical distances as the uniform distance

$$\|f - g\|_{C[a, b]} = R(f, g) = \sup \{|f(x) - g(x)| : x \in [a, b]\}$$

and the integral L_p $(1 \leq p \leq \infty)$ distances

$$\|f - g\|_{L_p[a,b]} = \{\frac{1}{b-a} \int_a^b |f(x) - g(x)|^p \, dx\}^{1/p}$$

are A-distances. The Hausdorff distance [2] is also an A-distance.

In order to investigate when from the discrete convergence of a sequence of functions follows a convergence in respect to a given A-distance, we'll introduce some characteristics of the functions, connected with this distance.

2. Baire Functions. We shall consider real functions f, defined in Ω. We'll suppose, that Ω is a compact metric space and the distance in Ω will be $\rho(x, t)$, $x, t \in \Omega$.

Let A_Ω be the set of all real and bounded functions, which are defined in Ω.

Let Ω' be an arbitrary subset of Ω and let d be the non-negative number

$$d = d(\Omega, \Omega') = \sup \{\inf \{\rho(x, x') : x' \in \Omega'\} : x \in \Omega\}.$$

That means, that if $\delta > d$, then for every $x \in \Omega$ there exists $x' \in \Omega'$, for which $\rho(x, x') < \delta$.

Definition 3. Upper (respectively lower) expanded Baire function for the function $f \in A_{\Omega'}$, for $\delta > d(\Omega, \Omega')$, we shall call the function from A_Ω

$$S(\delta, \Omega', f; x) = \sup\{f(t) : \rho(x, t) \leq \delta, t \in \Omega'\}, x \in \Omega,$$

(respectively $I(\delta, \Omega', f; x) = \inf\{f(t) : \rho(x,t) \leq \delta, t \in \Omega'\}, x \in \Omega$.

When Ω' coincides with Ω, instead of $S(\delta, \Omega, f)$ and $I(\delta, \Omega, f)$ we shall write $S(\delta, f)$ and $I(\delta, f)$.

From Definition 3 follows immediately, that for $\delta' > \delta > 0$ the inequalities

$$(1) \qquad I(\delta', f) \leq I(\delta, f) \leq f \leq S(\delta, f) \leq S(\delta', f)$$

are satisfied. Then, for every $x \in \Omega$ there exist the limits

$$(2) \qquad S(f; x) = \lim_{\delta \to +0} S(\delta, f; x), \quad I(f, x) = \lim_{\delta \to +0} I(\delta, f; x).$$

The functions $S(f)$ and $I(f)$ are the well-known upper and lower Baire functions for the function f. It is evident, that the operators S and I in A_Ω are projections, i. e.

$$(3) \qquad S^2 = S, \quad I^2 = I.$$

From (1) and (2) follows, that for every function $f \in A_\Omega$ and for every $\delta > 0$ the following inequalities are satisfied:

$$(4) \qquad I(\delta, f) \leq I(f) \leq f \leq S(f) \leq S(\delta, f).$$

It is easy to see, that the necessary and sufficient condition for the function $f \in A_\Omega$ to be continuous in the point $x_0 \in \Omega$ are the equalities

$$(5) \qquad S(f; x_0) = I(f; x_0) = f(x_0).$$

Lemma 1. *If $f, g \in A_{\Omega'}$, $\Omega' \subset \Omega$, and for every $x \in \Omega'$ the inequality $|f(x) - g(x)| < \lambda$ is satisfied, then for $\delta > d = d(\Omega, \Omega')$ we have $|S(\delta, f; x) - S(\delta, g; x)| \leq \lambda$ for every $x \in \Omega$, i. e.*

$$\| S(\delta, f) - S(\delta, g) \| \leq \lambda,$$

where $\|f\| = \sup\{|f(t)| : t \in \Omega\}$.

Proof. From the condition of the lemma we have that $f(t) \leq g(t) + \lambda$; $t \in \Omega'$. From this follows that if $x \in \Omega$, then for every $t \in \Omega'$, for which $\rho(x, t) < \delta$ (such a t exists, since $d(\Omega', \Omega) < \delta$), the next inequalities are satisfied

$$f(t) \leq \sup\{g(t) + \lambda : \rho(x, t) \leq \delta, t \in \Omega'\} = S(\delta, g; x) + \lambda$$

or

$$(6) \qquad S(\delta, f; x) \leq S(\delta, g; x) + \lambda.$$

Analogically we have

$$(7) \qquad S(\delta, g; x) \leq S(\delta, f; x) + \lambda.$$

From (6) and (7) we obtain the proof of the lemma.

3. Modulus of A-continuity. Let B_Ω be a linear subspace of A_Ω, which together with every function f contains also $S(f)$, $I(f)$, $S(\delta, f)$, $I(\delta, f)$ for every $\delta>0$. Let in B_Ω an A-distance r be defined. Then for every function f from the metric space (B_Ω, r) a modulus of A-continuity will be defined.

Definition 4. If $f \in (B_\Omega, r)$, then

$$\tau(f;\ \delta)_r = \tau(f,\ \delta) = r(S(\delta/2,\ f),\ I(\delta/2,\ f)),$$

$\delta>0$, will be called the modulus of A-continuity of the function f.

From the properties of the A-distance and the inequalities (1) immediately follows that for every function $f \in (B_\Omega, r)$ and for every $\delta'>\delta>0$ the inequalities

$$\tau(f;\ \delta') \geqq \tau(f;\ \delta) \geqq 0, \tag{8}$$

are valid, i. e. $\tau(f;\ \delta)$ is non-negative and non-decreasing function of δ for $\delta>0$.

From the inequalities (8) follows, that for every function $f \in (B_\Omega, r)$ there exists the limit

$$\lim_{\delta \to +0} \tau(f;\ \delta) = \mu \geqq 0.$$

Definition 5. The function $f \in (B_\Omega, r)$ will be called A-continuous in the metric space (B_Ω, r), if

$$\lim_{\delta \to +0} \tau(f;\ \delta) = 0.$$

From the Definition 4 and the inequalities (4) follows that the necessary and sufficient condition for the function $f \in (B_\Omega, r)$ to be A-continuous, is the equality

$$r(S(f),\quad I(f)) = 0. \tag{9}$$

This condition may be described as follows. For the function f to be A-continuous in respect to the A-distance r, it is necessary and sufficient that its upper and lower Baire functions be non-distinguishable in respect to this distance.

If the function f is A-continuous in respect to the distance r, then the same property is possessed by the functions $S(f)$ and $I(f)$.

Lemma 2. *Let $f,\ g \in (B_\Omega, r)$ and $\Omega' \subset \Omega$, $d = d(\Omega, \Omega')$. If for every $x \in \Omega'$ the inequality $|f(x) - g(x)| \leqq \lambda$ holds, then for every $\delta > d$ we have*

$$r(f,\ g) \leqq \tau(f;\ 2\delta) + \tau(g,\ 2\delta) + \lambda. \tag{10}$$

Proof. We shall use the obvious inequalities

$$\begin{aligned} &I(\delta,\ f) \leqq f \leqq S(\delta,\ \Omega',\ f) \leqq S(\delta,\ f),\\ &I(\delta,\ g) \leqq g \leqq S(\delta,\ \Omega',\ g) \leqq S(\delta,\ g) \end{aligned} \tag{11}$$

and the Lemma 1, according to which

$$r(S(\delta,\ \Omega',\ f),\quad S(\delta,\ \Omega',\ g)) \leqq \| S(\delta,\ \Omega',\ f) - S(\delta,\ \Omega',\ g) \| \leqq \lambda. \tag{12}$$

The inequality $r(f,\ g) \leqq \| f - g \|$ follows from the definition of the A-distance.

From the inequalities (11) and (12), using the properties of the A-distance, we have

$$r(f, g) \leqq r(f, S(\delta, \Omega', f)) + r(S(\delta, \Omega', f), S(\delta, \Omega', g))$$
$$+ r(S(\delta, \Omega', g), g) \leqq r(I(\delta, f), S(\delta, f)) + \lambda$$
$$+ r(I(\delta, g), S(\delta, g)) = \tau(f; 2\delta) + \tau(g; 2\delta) + \lambda.$$

The lemma is proved.

4. Conditions for Convergence. We are ready to formulate the basic theorem, which contains a sufficient condition, in which from a discrete convergence follows a convergence in respect to the distance in a metric space with A-distance.

Definition 6. A set of functions $\{f_\alpha\}$, $f_\alpha \in (B_\Omega, r)$ will be called A-equicontinuous, if there exists a non-negative and non-decreasing function $\tau(\delta)$ for $\delta > 0$ and $\lim_{\delta \to +0} \tau(\delta) = 0$, such that for every function f_α from $\{f_\alpha\}$ the inequality $\tau(f_\alpha; \delta) \leqq \tau(\delta)$ holds for every $\delta > 0$.

Theorem 1. *Let $\{f_n\}_1^\infty$ be a sequence of A-equicontinuous functions belonging to (B_Ω, r) and $f \in (B_\Omega, r)$. If the function f is A-continuous, then from the discrete convergence of the sequence $\{f_n\}_1^\infty$ to f follows the convergence with respect to the distance r, i. e.* $\lim_{n\to\infty} r(f_n, f) = 0$.

Proof. From the Lemma 2 and the discrete convergence of the sequence $\{f_n\}_1^\infty$ to f follows, that for every $\delta > 0$ there exists a cardinal N such that for $n \geqq N$ the inequality

$$r(f_n, f) \leqq \tau(f_n; 2\delta) + \tau(f; 2\delta) + \delta$$

holds.

Because the functions f_n are A-equicontinuous and the function f is A-continuous, then for every $\varepsilon > 0$ there exists such a $\delta > 0$ that $\tau(f_n; 2\delta) + \tau(f; 2\delta) + \delta < \varepsilon$. The theorem is proved.

5. Examples. We shall consider the modulus τ for A-continuity in some concrete spaces.

If for the distance in (B_Ω, r) we take the uniform distance $R(f, g) = \|f - g\|_C$, then

$$\tau(f; \delta)_C = \|S(\delta/2, f) - I(\delta/2, f)\|_C = \sup\{|f(x) - f(t)| : \rho(x, t) \leqq \delta\} = \omega(f; \delta)_C,$$

i. e. the τ-modulus in this case is the well-known modulus of continuity.

It happens, that in the integral L_p-distances the τ-moduli are considerably different from the well-known moduli of continuity in L_p

$$\omega(f; \delta)_{L_p} = \sup_{|t| \leqq \delta} \{\frac{1}{b-a} \int_a^b |f(x+t) - f(x)|^p \, dx\}^{1/p}.$$

It is obvious, that the inequality

$$\omega(f; \delta)_{L_p} \leqq \tau(f; \delta)_{L_p} \tag{13}$$

always holds.

When the A-distance is generated by a norm, as in the case of the uniform distance and the L_p-distances, the τ-modulus can be defined through the norm of the usual local modulus of continuity

$$\omega(f, x; \delta) = \sup\{|f(t+h) - f(t)| : t, t+h \in [x - \delta/2, x + \delta/2] \cap \Omega\}.$$

It is easy to see, that

$$\tau(f;\ \delta)_{L_p}=\|\omega(f,\ \cdot\ ;\ \delta)\|_{L_p}.$$

The last equality gives us the possibility to define τ-moduli of higher order.

We shall restrict ourselves to the case, when Ω is an interval. Then the local modulus of continuity of the order k will be

$$\omega_k(f,\ x;\ \delta)=\sup\{|\Delta_h^k f(x)|: t,\ t+kh\in[x-k\delta/2,\ x+k\delta/2]\cap\Omega\},$$

where $\Delta_h^k f(x)$ is the k-th difference of f

$$\Delta_h^k f(x)=\sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}f(x+ih).$$

Then the k-th τ-modulus in L_p is defined as follows:

$$\tau_k(f;\ \delta)_{L_p}=\|\omega_k(f,\ \cdot\ ;\ \delta)\|_{L_p}.$$

Many applications of the τ-moduli of different order have shown that they are very useful for the estimation of the approximation in different situations. They are especially useful for estimations of the errors in numerical methods. This fact is not accidental. It is based on Theorem 1, which gives the conn ection between the discrete convergence and the convergence in respect to the distance.

6. The Properties of τ-moduli in L_p. We are giving without proofs some properties of the τ-moduli in L_p. The proofs can be found in the recent publications [3, 4].

The inequality (13) is valid for every cardinal k:

1) $\omega_k(f;\ \delta)_{L_p}\leq\tau_k(f;\ \delta)_{L_p}$.

Besides this, the next inequalities hold

2) $\tau_k(f+g;\ \delta)_{L_p}\leq\tau_k(f;\delta)_{L_p}+\tau_k(g;\ \delta)_{L_p}$;

3) $\tau_k(f;\ \delta)_{L_p}\leq\delta\tau_{k-1}(f';\ \frac{k}{k-1}\ \delta)_{L_p},\ k\geq 2$;

4) $\tau_1(f;\ \delta)_{L_p}\leq\delta\|f'\|_{L_p}$;

5) $\tau_k(f;\ m\delta)_{L_p}\leq(2m)^{2k+1}\tau_k(f;\ \delta)_{L_p},\ m$ — natural.

The inequalities 1)—5) allow to work with the τ-moduli as with the usual moduli of continuity.

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Received on June 20, 1981