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HIGH-ORDER APPROXIMATION WITH CONVOLUTION OPERATORS

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Summary. In this paper a class of convolution operators U_ρ of a certain type (1) is considered. It is proved that there exists a β satisfying conditions (i)—(iv) such that the order of approximation to $f(x)$ (f bounded and continuous on R) by $(U_\rho f)(x)$ at a point x, where $f''(x)$ exists, is equal to $\exp(-\rho/\alpha(\rho))$, given in Section 7.

1. Introduction. The convolution operators U_ρ, studied in this paper, are defined on the class M of all those real functions f, defined on the real axis R, that are bounded and continuous on R. The class B consists of all real functions β, defined on R and having the following four properties:

(i) $\beta \geqq 0$ on R;
(ii) $\beta(0)=1$, β is continuous at $t=0$;
(iii) for each $\delta>0$ is $\sup\{\beta(t):|t|\geqq\delta\}<1$;
(iv) $\beta \in L_1(R)$.

Then the operators U_ρ are defined on M by

$$(U_\rho f)(x)=I_\rho^{-1}\int_{-\infty}^{\infty} f(x-t)\,\beta^\rho(t)\,dt \quad (f\in M,\ \beta\in B,\ x\in R,\ \rho\geqq 1). \tag{1}$$

I_ρ is a normalizing factor:

$$I_\rho=\int_{-\infty}^{\infty}\beta^\rho(t)dt \quad (\rho\geqq 1). \tag{2}$$

It follows from (1) and (2) that $(U_\rho 1)(x)\equiv 1$.

Operators of this type, though of a very special nature, were considered a. o. by W e i e r s t r a s s [13], L a n d a u [4] and, somewhat more general than these, by K o r o v k i n [3]. Concerning the speed, with which the approximation

$$(U_\rho f)(x)\rightarrow f(x) \tag{3}$$

takes place, first results were given by B o j a n i c and S h i s h a [1], but only for a small subclass of operators of type (1), (2). They used the modulus of continuity of f. For the general class of operators U_ρ, defined above, results in this direction were derived by the author [5, 6] in sharp form.

By deriving a formula of Voronovskaya type the author showed in [8], using a β belonging to a subclass of B, that the order of approximation in (3) at a point x, where $f''(x)$ exists, can be $\rho^{-\alpha}$ $(\alpha>0)$, generalizing some results by him and Rathore [7]. By using another subclass of B he proved [9] that the order of approximation at such a point x can be exp $(-\sqrt{\rho}+p(\rho))$, where $p(\rho)=o(\sqrt{\rho})$ is a polynomial in some fractional power of ρ. In [10] this result was generalized to the existence of a β such that the order of approximation is $\exp(\rho^{1/p}+q(\rho))$, $p>0$, $q(\rho)=o(\rho^{1/p})$ being a polynomial in some smaller power of ρ. As a counterpart the author showed [11], that the approximation can be very slow at such an x, viz. $(\log_n \rho)^{-1}$ $(n\in N)$.

The aim of the present paper is to show that there exists to each $n\in N$ a $\beta\in B$ such that if $f''(x)$ exists at a point $x\in R$ and $f'(x)\neq 0$, (3) can be written as

$$(4)\quad (U_\rho f)(x)-f(x)=-\frac{1}{2}\exp\{-\rho\log 2/(\log\rho\log_2\rho\ldots\log_{n-1}\rho\log_n^2\rho)\times(1+o(1))\}\,f'(x)(1+o(1)),$$

so that the order of approximation at x is $\exp\{-\rho\log 2/(\log\rho\log_2\rho\ldots\log_{n-1}\rho\log_n^2\rho).\ (1+o(1))\}$. This formula may hold at all points of a finite interval. This improves a result by Totik [12], who showed by constructing an appropriate stepfunction $\beta\in B$, that the order of approximation can be $\exp(-\rho^{\sigma(x)})$ with $\sigma(x)=1-(\log\log\rho)^{-1/k}$ $(k>1)$, which, however, is worse than our result. On the other hand, Totik shows that only in some very special cases of f (not of β) there exists a $\beta\in B$ such that the order of approximation is exp $(-\sigma x)$ $(\sigma>0$, σ constant). Moreover he conjectures, that if $\alpha(\rho)>0$ $(\rho\geqq 1)$, $\alpha(\rho)\to\infty$, no matter how slowly, if $\rho\to\infty$, there exists a $\beta\in B$ such that the order of approximation in the above sense is $\exp(-\rho/\alpha(\rho))$. Hence our result strengthens the believe that some day this unsolved problem will be solved.

2. The Difference $(U_\rho f)(x)-f(x)$. In the following e_n $(n\in N,\ n\geqq 2)$ will stand for exp (exp (. . . (exp 1) . . .), where exp is occurring n times; $\log_n a$ $(n\in N,\ n\geqq 2,\ a>e_{n-2})$ will stand for $\log\log\ldots\log a$, where log is occurring n times.

Let $\delta=e_7^{-1}$. The function $\beta(t)$ is defined on R in the following way:

$$(5)\quad \begin{aligned}
&\beta(0)=1,\\
&\beta(t)=1-(\log_3(1/t))^{-1} && (0<t\leqq\delta),\\
&\beta(t)=\alpha e^{-t^2} && (t>\delta),\\
&\beta(t)=1-(\log_4(1/|t|)^{-1}) && (-\delta\leqq t<0),\\
&\beta(t)=\alpha' e^{-t^2} && (t<-\delta),
\end{aligned}$$

α, α' being chosen such that β is continuous at $t=\pm\delta$. Obviously, $\beta\in B$. Since β is continuous on R, a general theorem [8, Theorem 1] learns that (3) holds at all points of R. In order to investigate the speed, with which the approximation in (3) takes place, it is assumed that at an arbitrarily chosen and then fixed point x the second derivative $f''(x)$ exists. Then Taylor's expansion around x reads

$$(6)\quad f(x-t)-f(x)=-t\,f'(x)+t^2 f''(x)/2+t^2\gamma_x(t)\qquad (t\in R).$$

Setting $\gamma_x(0)=0$, then $\gamma_x(t)$ is bounded and continuous on R. Defining for $\rho\geqq 1$

$$(7)\qquad I_{\nu\rho}(\delta)=\int_{-\delta}^{\delta} t^\nu \beta^\rho(t)\,dt,\quad J_{\nu\rho}(\delta)=\int_{|t|\geqq\delta} t^\nu \beta^\rho(t)\,dt\quad(\nu=0,1,2),$$

$$(8)\qquad K_\rho=\int_{-\infty}^{\infty} t^2\gamma_x(t)\,\beta^\rho(t)\,dt,$$

it follows from (6), (7) and (8) that, if $\rho\geqq 1$,

$$(9)\qquad (U_\rho f)(x)-f(x)=I_\rho^{-1}\{-I_{1\rho}(\delta)f'(x)+I_{2\rho}(\delta)f''(x)/2-J_{1\rho}(\delta)f'(x)$$
$$+J_{2\rho}(\delta)f''(x)/2+K_\rho\}.$$

In view of the first formula in (7) the following functions are introduced:

$$(10)\qquad A_{\nu\rho}(\delta)=\int_0^\delta t^\nu\beta^\rho(t)\,dt,\quad B_{\nu\rho}(\delta)=\int_0^\delta t^\nu\beta^\rho(-t)\,dt\quad(\rho=0,1,2),$$

so that

$$(11)\qquad I_{\nu\rho}(\delta)=A_{\nu\rho}(\delta)+(-1)^\nu B_{\nu\rho}(\delta)\qquad(\nu=0,1,2).$$

The asymptotic behaviour of $A_{\nu\rho}(\delta)$ and $B_{\nu\rho}(\delta)$, if $\rho\to\infty$, is now of interest.

3. The Function $\lambda(\rho,u)$. In this section the function

$$(12)\qquad \lambda(\rho,u)=\gamma^{-1}u-\rho\log(1-(\log_2 u)^{-1})\quad(0<\gamma\leqq 1,\ \rho\geqq 1,\ u\geqq e_6)$$

will be investigated with emphasis on its asymptotic behaviour, if $\rho\to\infty$. First of all

$$(13)\qquad d\lambda/du=\gamma^{-1}(1-(u\log u\,.\log_2 u\,.(\log_2 u-1))^{-1}\sigma)\quad(\sigma=\gamma\rho),$$

$$(14)\qquad d^2\lambda/du^2=\gamma^{-1}\sigma(\log u\,.\log_2 u\,.(\log_2 u-1)+\log_2 u\,.(\log_2 u-1)$$
$$+2\log_2 u-1)(u\log u\,.\log_2 u\,.(\log_2 u-1))^{-2}.$$

From (14) it follows that $d^2\lambda/du^2>0$ $(u\in[e_6,\infty))$. Consequently, $\lambda(\rho,u)$ is strict convex on $[e_6,\infty)$. This means that the equation $d\lambda/du=0$, i. e.

$$(15)\qquad u\log u\,.\log_2 u\,(\log_2 u-1)=\sigma$$

has at most one root on $[e_6,\infty)$. As $d\lambda/du<0$ at e_6 for all sufficiently large values of σ (say $\sigma\geqq\sigma_1$) and $d\lambda/du>0$ if u is then sufficiently large, there is exactly one solution $u=u_0$ of (15) on $[e_6,\infty)$, if $\sigma\geqq\sigma_1$. Clearly a first approximation to u_0 is $u_1=\sigma/(\log\sigma\ \log_2^2\sigma)$.

In order to get a second approximation the substitution $u=u_1(1+\varphi(\sigma))$ $(\varphi(\sigma)=o(1))$ in (15) is made, which yields

$$(16)\quad u_0=\sigma(\log\sigma\log_2^2\sigma)^{-1}\{1+1/(\log_2\sigma)+\log^{-1}\sigma(\log_2\sigma+2\log_3\sigma)(1+o(1))\}.$$

In this way it is possible to construct more terms between the brackets. However, it will now be proved that the solution u_0 of (15), belonging to $[e_6,\infty)$, can be written in the form

$$(17)\quad u_0=(\log\sigma\ \log_2^2\sigma)^{-1}\sigma P(1/(\log_2\sigma),\ 1/(\log\sigma),\ (\log_2\sigma+2\log_3\sigma)/(\log\sigma)),$$

in which $P(\mu_1,\mu_2,\mu_3)$ is a power series in the three variables μ_1,μ_2,μ_3, convergent for all sufficiently small values of them (Cf. [2, §2.4]). In fact, substitution in (15) of $u=\sigma v/(\log\sigma\log_2^2\sigma)$ and setting

(18) $\qquad 1/(\log_2 \sigma)=\mu_1,\ 1/(\log \sigma)=\mu_2,\ (\log_2 \sigma+2 \log_3 \sigma)/(\log \sigma)=\mu_3,$

leads to the following equation for v:

(19)
$$v\{1-\mu_3+\mu_2 \log v\}\ \{1+\mu_1 \log (1-\mu_3+\mu_2 \log v)\}$$
$$\times\{1-\mu_1+\mu_1 \log (1-\mu_3+\mu_2 \log v)\}=1.$$

Substitution in this equation of

(20)
$$v=e^w$$

and setting

(21)
$$G(w)=1-(1-\mu_3+\mu_2 w)\ \{1+\mu_1 \log (1-\mu_3+\mu_2 w\}$$
$$\times\{1-\mu_1+\mu_2 \log(1-\mu_3+\mu_2 w)\},$$

yields for w the equation

(22)
$$F(w)=e^{-w}-1+G(w)=0.$$

From (21) it follows that there exists a positive number $\varkappa$ such that for all μ_1, μ_2, μ_3, w satisfying

(23)
$$|\mu_1|\leqq\varkappa,\quad |\mu_2|\leqq\varkappa,\quad |\mu_3|\leqq\varkappa,\quad |w|=\pi,$$

the inequalities

(24)
$$|\mu_3-\mu_2 w|\leqq 2^{-1}$$

and

(25)
$$|G(w)|\leqq 4^{-1}\ \min\ \{|e^{-w}-1|:|w|=\pi\}$$

hold. As $e^{-w}-1=0$ has exactly one solution w with $|w|\leqq\pi$, viz. $w=0$, it follows from Rouché's theorem that the equation (22) has exactly one solution $w=w_0$ inside the circle $|w|=\pi$. Cauchy's theorem then gives w_0 as

(26)
$$w_0=\frac{1}{2\pi i}\int_{|w|=\pi}\frac{w F'(w)}{F(w)}\,dw,$$

where the path of integration is taken in the positive direction. Because of (22) and (25), $(F(w))^{-1}$ can be developed on $|w|=\pi$ in the following way:

(27)
$$(F(w))^{-1}=(e^{-w}-1)^{-1}\left\{1+\frac{G(w)}{e^{-w}-1}\right\}^{-1}=\sum_{k=0}^{\infty}(-1)^k(e^{-w}-1)^{-k-1}(G(w))^k.$$

Writing under conditions (23)
$$\log(1-\mu_3+\mu_2 w)=\sum_{j=1}^{\infty}(-1)^j j^{-1}(-\mu_3+\mu_2 w)^j$$
$$=\sum_{j=1}^{\infty} j^{-1}\sum_{m=0}^{j}\binom{j}{m}(\mu_2 w)^m(-1)^m \mu_3^{j-m},$$

substitution of this result in (21) and then (21) in (27) gives
$$(F(w))^{-1}=\sum_{k=0}^{\infty}(-1)^k(e^{-w}-1)^{-k-1}[1-(1-\mu_3+\mu_2 w)$$
$$\times\{1+\mu_1\sum_{j=1}^{\infty}j^{-1}\sum_{m=0}^{j}\binom{j}{m}(\mu_2 w)^m(-1)^m\mu_3^{j-m}\}$$

$$\times\{1-\mu_1+\mu_1\sum_{n=1}^{\infty} n^{-1}\sum_{p=0}^{n}\binom{n}{m}(\mu_2 w)^p(-1)^p\mu_3^{n-p}\}]^k$$

$$=\sum_{k=0}^{\infty}(-1)^k(e^{-w}-1)^{-k-r-s-t-1}\sum_{r,s,t=0}^{\infty} d_{krst}\,\mu_1^r\mu_3^t(\mu_2 w)^s,$$

where all series are converging absolutely and uniformly, if (23) are satisfied. Hence (26) gives w_0 in the form of a triple power series

$$w_0=\sum_{r,s,t=0}^{\infty} f_{rst}\,\mu_1^r\mu_2^s\mu_3^t,$$

converging, if μ_1, μ_2, μ_3 satisfy (23). Returning to (20), $v_0=e^{w_0}$ can be written in the form of such a triple power series:

$$v_0=\sum_{r,s,t=0}^{\infty} g_{rst}\,\mu_1^r\mu_2^s\mu_3^t,$$

μ_1, μ_2, μ_3 satisfying (23). A few coefficients are $g_{000}=g_{100}=g_{010}=g_{001}=1$, $g_{110}=-1$, $g_{101}=2$, $g_{111}=0$, $g_{200}=1$, $g_{210}=-1.5$, $g_{211}=2$.

Recalling μ_1, μ_2, μ_3 were given in fact by (18), the formula (17) is proved.

Returning now to (16) and using for the sake of brevity the notations (18), it follows from (16) that

$$\log u_0=\log\sigma\,.\,\{1-\mu_3+o(\mu_3)\};$$

(28) $$\log_2 u_0=\log_2\sigma\,.\,\{1-\mu_1\mu_3+o(\mu_1\mu_3)\};$$

$$\log_2 u_0-1=\log_2\sigma\,.\,\{1-\mu_1-\mu_1\mu_3+o(\mu_1\mu_3)\}.$$

$\lambda(\rho,u)$ has a minimum at $u=u_0$ ($\sigma\geqq\sigma_1$) (as was noticed before, this minimum is the only extremum on $[e_6,\infty)$ if $\sigma\geqq\sigma_1$). Further $\lambda(\rho,u_0)=\mu_1\rho+1/2\,\mu_1^2\rho\,(1+o\,(1))$. Hence

(29) $$\exp(-\lambda(\rho,u_0))=\exp(-\mu_1\rho+1/2\,\mu_1^2\rho\,(1+o\,(1)))$$

and this value is the maximum value of $e^{-\lambda(\rho,u)}$ on $[e_6,\infty)$, if $\rho\geqq\gamma^{-1}\sigma_1$.

4. Asymptotic Behaviour of $A_{\nu\rho}(\delta)\,(\nu=0,1,2)$. Using definition (5) of $\beta(t)$, $A_{\nu\rho}(\delta)$, given by (10), can be written after having made the substitution $t=e^{-u}$, as

$$A_{\nu\rho}(\delta)=\int_{e_6}^{\infty} e^{-(\nu+1)u}\{1-(\log_2 u)^{-1}\}^{\rho}\,du.$$

This is given in the form

(30) $$A_{\nu\rho}(\delta)=\int_{e_6}^{\infty} e^{-\lambda(\rho,u)}\,du,\quad \gamma=(\nu+1)^{-1},$$

in which $\lambda(\rho,u)$ is the function, defined and studied in the previous section. Let $\mu(\sigma)$ be given by

(31) $$\mu(\sigma)=\sigma\log_3\sigma/\log\sigma\log_2^3\sigma\quad(\sigma\geqq\sigma_2),$$

where σ_2 is chosen so large, that $\sigma_2\geqq\sigma_1$ and $u_0-\mu(\sigma)>e_6$ for all $\sigma\geqq\sigma_2$. Then the integral in (31) is split up in the following way:

$$A_{\nu\rho}(\delta)=\{\int_{e_6}^{u_0-\mu(\sigma)}+\int_{u_0-\mu(\sigma)}^{u_0+\mu(\sigma)}+\int_{u_0+\mu(\sigma)}^{\infty}\}\,e^{-\lambda(\rho,u)}\,du=I_1+I_2+I_3. \tag{32}$$

The integrals I_1, I_2 and I_3 are investigated separately.

Investigation of I_2. On the interval $J_\mu(u_0)=[u_0-\mu(\sigma), u_0+\mu(\sigma)]$ is

$$\lambda(\rho, u)=\lambda(\rho, u_0)+(u-u_0)\frac{d\lambda}{du_0}+\frac{1}{2}(u-u_0)^2\frac{d^2\lambda}{du_1^2}\quad (u_1\in J_\mu(u_0)) \tag{33}$$

with $d\lambda/du_0=0$. Substitution of (16) in (14) yields

$$\frac{d^2\lambda}{du_0^2}=(\gamma\sigma)^{-1}\log\sigma\,\log_2^2\sigma(1+o(1))>0 \tag{34}$$

and it follows that for all $u\in J_\mu(u_0)$, $(d^2\lambda/du^2)=(d^2\lambda/du_0^2)(1+o(1))$, $o(1)$ being uniform in $u\in J_\mu(u_0)$. This means that

$$I_2=\exp(-\lambda(\rho,u_0))\int_{u_0-\mu(\sigma)}^{u_0+\mu(\sigma)}\exp(\lambda(\rho,u_0)-\lambda(\rho,u))\,du$$

$$=\exp(-\lambda(\rho,u_0))\int_{u_0-\mu(\sigma)}^{u_0+\mu(\sigma)}\exp(-1/2\,(u-u_0)^2\frac{d^2\lambda}{du_0^2}(1+o(1)))\,du,$$

$o(1)$ uniform in $u\in J_\mu(u_0)$. Substitution of $(u-u_0)(2^{-1}d^2\lambda/du_0^2)^{1/2}=t$ is possible because of (34) and setting

$t_1=-\mu(\sigma)(2^{-1}d^2\lambda/du_0^2)^{1/2}$, $t_2=\mu(\sigma)(2^{-1}d^2\lambda/du_0^2)^{1/2}$ it gives

$$I_2=\exp(-\lambda(\rho,u_0))\,2^{1/2}(d^2\lambda/du_0^2)^{-1/2}\int_{t_1}^{t_2}\exp(-t^2(1-o(1)))\,dt.$$

As $t_1\to-\infty$, $t_2\to\infty$ if $\rho\to\infty$, it follows that

$$I_2=(\log\sigma\log_2^2\sigma)^{-1/2}(2\pi\gamma\sigma)^{1/2}e^{-\lambda(\rho,\mu_0)}(1+o(1)). \tag{35}$$

Hence, by (29) and (18)

$$I_2=(\log\sigma\log_2^2\sigma)^{-1/2}(2\pi\gamma\sigma)^{1/2}[\exp\{-\sigma(\log_2\sigma)^{-1}-\sigma(2\log_2^2\sigma)^{-1}\times(1+o(1))\}](1+o(1))\quad(\rho\to\infty). \tag{36}$$

Investigation of I_3. We write

$$I_3=\int_{u_0+\mu(\sigma)}^{\infty}e^{-\lambda(\rho,u)}\,du=\exp(-\lambda(\rho,u_0)) \tag{37}$$

$$\times\exp(\lambda(\rho,u_0)-\lambda(\rho,u_0+\mu(\sigma)))\int_{u_0+\mu(\sigma)}^{\infty}\exp(\lambda(\rho,u_0+\mu(\sigma))-\lambda(\rho,u))\,du$$

and consider first the factor $\exp(\lambda(\rho,u_0)-\lambda(\rho,u_0+\mu(\sigma)))$. Using definition (12) of $\lambda(\rho,u)$, it is clear that for sufficiently large values of σ, $\sigma\geqq\sigma_3$ $(\sigma_3\geqq\sigma_2)$,

$$\lambda(\rho,u_0)-\lambda(\rho,u_0+\mu(\sigma))=-\gamma^{-1}\mu(\sigma)+\gamma^{-1}\sigma\{1/(\log_2u_0)-1/(\log_2(u_0+\mu(\sigma)))+1/(\log_2^2u_0)-1/(\log_2^2(u_0+\mu(\sigma)))\}$$

$$<\gamma^{-1}\{-\mu(\sigma)+\sigma(1/(\log_2u_0)-1/(\log_2(u_0+\mu(\sigma))))(1+1/(2\log_2u_0))\}$$

$$\leqq \gamma^{-1}\{-\mu(\sigma)+\sigma(1/(\log_2 u_0)-(\log_2 u_0+\mu(\sigma)/((u_0 \log u_0))^{-1}(1+1/(2\log_2 u_0))\}$$
$$=\gamma^{-1}\{-\mu(\sigma)+\sigma(1/(\log_2 u_0)-1/(\log_2 u_0).(1+\mu(\sigma)/(u_0\log u_0\log_2 u_0))^{-1}) \times(1+1/(2\log_2 u_0))\}$$
$$\leqq \gamma^{-1}\{-\mu(\sigma)+\sigma\mu(\sigma)/(u_0\log u_0\log_2 u_0).(1+1/(2\log_2 u_0))\}$$
$$\leqq \gamma^{-1}\mu(\sigma)\{-1+\sigma/(u_0\log u_0\log_2 u_0).(1+1/(2\log_2 u_0))\}.$$

Using (16) and (28), this yields that for all sufficiently large values σ, $\sigma\geqq\sigma_4 \geqq\sigma_3$ say,

$$\lambda(\rho, u_0)-\lambda(\rho, u_0+\mu(\sigma))\leqq -\gamma^{-1}\mu(\sigma)\{1/(\log_2\sigma)-2(\log_2\sigma)/(\log\sigma)\} \times(1+1/(2\log_2 u_0)).$$

Hence there exists a $\sigma_5\geqq\sigma_4$ such that

$$(38)\qquad \lambda(\rho, u_0)-\lambda(\rho, u_0+\mu(\sigma))\leqq -\mu(\sigma)/(2\gamma\log_2\sigma)\quad(\sigma\geqq\sigma_5).$$

Next, the integral in the right-hand side of (37) is investigated. For all sufficiently large values of σ, $\sigma\geqq\sigma_6$ ($\sigma_6\geqq\sigma_5$) say, is

$$\lambda(\rho, u_0+\mu(\sigma))-\lambda(\rho, u_0+\mu(\sigma)+v)\leqq -\gamma^{-1}v+\rho\{1/(\log_2(u_0+\mu(\sigma))) -1/(\log_2(u_0+\mu(\sigma)+v))+1/(\log_2^2(u_0+\mu(\sigma)))-1/(\log_2^2(u_0+\mu(\sigma)+v))\}$$
$$=-\gamma^{-1}v+\rho\left\{\frac{\log_2(u_0+\mu(\sigma)+v)-\log_2(u_0+\mu(\sigma))}{\log_2(u_0+\mu(\sigma))\log_2(u_0+\mu(\sigma)+v)}\right.$$
$$\left.+\frac{\log_2^2(u_0+\mu(\sigma)+v)-\log_2^2(u_0+\mu(\sigma))}{\log_2^2(u_0+\mu(\sigma))\log_2^2(u_0+\mu(\sigma)+v)}\right\}.$$

Using the mean value theorem of the differential calculus to both the fractions between curled brackets shows the existence of v_1 and v_2 between 0 and v such that the expression between the curled brackets is equal to

$$\frac{v\dfrac{d}{dt}(\log_2 t)_{u_0+\mu(\sigma)+v_1}}{\log_2(u_0+\mu(\sigma))\log_2(u_0+\mu(\sigma)+v)}+\frac{v\dfrac{d}{dt}(\log_2 t)_{u_0+\mu(\sigma)+v_2}}{\log_2^2(u_0+\mu(\sigma))\log_2^2(u_0+\mu(\sigma)+v)}.$$

Hence

$$(39)\quad \lambda(\rho, u_0+\mu(\sigma))-\lambda(\rho, u_0+\mu(\sigma)+v)\leqq -\gamma^{-1}v+\rho v\{1/((u\log u\log_2 u)_{u_0+\mu(\sigma)}) +(2\log_2(u_0+\mu(\sigma)+v))/((u\log u\log_2 u)_{u_0+\mu(\sigma)}.\log_2^2(u_0+\mu(\sigma)+v))\}$$
$$\leqq \gamma^{-1}v\{-1+\sigma/((u_0+\mu(\sigma))\log u_0\log_2^2 u_0).(1+2/(\log_2 u_0))\}$$
$$\leqq \gamma^{-1}v\{-1+\sigma/(u_0\log u_0\log_2^2 u_0).(1-(\mu(\sigma))/u_0+(\mu_0^2(\sigma))/u_0^2)(1+2/(\log_2 u_0))\}.$$

Using (16), (28) and (31), it follows from (39) that for all sufficiently large values of σ, $\sigma\geqq\sigma_7$ ($\sigma_7\geqq\sigma_6$) say

$$(40)\qquad \lambda(\rho, u_0+\mu(\sigma))-\lambda(\rho, u_0+\mu(\sigma)+v)\leqq -v/2\gamma.(\log_3\sigma)/(\log_2\sigma).$$

Finally, it follows from (37), (38) and (40) that for $\sigma\geqq\sigma_7$

(41) $$I_3 \leqq e^{-\lambda(\rho, u_0)} . \exp(-(\mu(\sigma))/(2\gamma \log_2 \sigma)) . \int_0^\infty \exp(-v/2\gamma . (\log_3 \sigma)/(\log_2 \sigma))\, dv$$
$$= e^{-\lambda(\rho, u_0)} (2\gamma \log_2 \sigma)/(\log_3 \sigma) . \exp(-(\mu(\sigma))/(2\gamma \log_2 \sigma)).$$

Comparing this result with (36) and using (31), it is clear that

(42) $$I_3 = o(I_2) \quad (\rho \to \infty).$$

Investigation of I_1. Let the function $\mu^* = \mu^*(\sigma)$ be defined by

(43) $$\mu^* = \sigma/(\log \sigma \log_2^2 \sigma) \quad (\sigma \geqq \sigma_2).$$

Then $\mu^* \in [u_0 - \mu(\sigma), u_0)$. I_1 is written as

(44) $$I_1 = \int_{e_6}^{u_0-\mu(\sigma)} e^{-\lambda(\rho, u)}\, du = e^{-\lambda(\rho, u^*)} . \exp(\lambda(\rho, u^*) - \lambda(\rho, u^* - \mu(\sigma)))$$
$$\times \int_{e_6}^{u_0-\mu(\sigma)} \exp(\lambda(\rho, u^* - \mu(\sigma)) - \lambda(\rho, u))\, du.$$

Hence, because $\lambda(\rho, u)$ attains its minimum at $u = u_0$ and $u^* \neq u_0$,

(45) $$I_1 < e^{-\lambda(\rho, u_0)} . \exp(\lambda(\rho, u^*) - \lambda(\rho, u^* - \mu(\sigma))) \int_{e_6}^{u_0-\mu(\sigma)} \exp(\lambda(\rho, u^*$$
$$- \mu(\sigma)) - \lambda(\rho, u))\, du.$$

Then it can be proved in a similar way as (38) was derived, that

(46) $$\lambda(\rho, u^*) - \lambda(\rho, u^* - \mu(\sigma)) \leqq -\frac{1}{2\gamma} \mu(\sigma) \;(\log_2 \sigma)/(\log \sigma).$$

Again, as $u^* - \mu(\sigma) < u_0 - \mu(\sigma)$ and $\lambda(\rho, u)$ is decreasing on $[e_6, u_0]$, it is obvious that the integral in the right-hand side of (45) is smaller than u_0. Using this and (46) in (45), it follows that

$$I_1 < e^{-\lambda(\rho, u_0)} u_0 \exp(-(\mu(\sigma))/(2\gamma) . (\log_2 \sigma)/(\log \sigma)).$$

Because of (16), (31) and (36) this results in

(47) $$I_1 = o(I_2) \quad (\rho \to \infty).$$

Then it follows from (32), (47) and (42) that $A_{\nu\rho}(\delta) = I_2(1 + o(1))$ as $\rho \to \infty$. Hence, in view of (36) and recalling that $\sigma = \gamma\rho$, this gives, if $\nu = 0, 1, 2$,

(48) $$A_{\nu\rho}(\delta) = ((2\pi\gamma\sigma)/(\log \sigma \log_2^2 \sigma))^{1/2} . \exp(-\rho/(\log_2 \sigma) - \rho/(2 \log_2^2 \sigma)(1 + o(1)))$$
$$\times (1 + o(1))$$
$$= (\nu+1)^{-1} ((2\pi\rho)/(\log \rho \log_2^2 \rho))^{1/2} \exp\{-\rho(1/(\log_2 \rho) + (\log \gamma / \log \rho \log_2^2 \rho)$$
$$\times (1 + o(1)))\} . (1 + o(1)) \quad (\rho \to \infty).$$

5. Asymptotic Behaviour of $(U_\rho f)(x) - f(x)$. From the definition (5) of $\beta(t)$ it can be shown on the same lines as those above that (see Section 6)

(49) $$B_{\nu\rho}(\delta) = O\{\rho \exp(-\rho/(2(\nu+1))(\log_3 \sigma)^{-1})\} \quad (\rho \to \infty).$$

From (48) and (49) it then follows that

(50) $$I_{\nu\rho}(\delta) = A_{\nu\rho}(\delta)(1 + o(1)) \quad (\nu = 0, 1, 2;\ \rho \to \infty).$$

Thus, $I_\rho = I_{0\rho}(\delta)(1+o(1)) = A_{0\rho}(1+o(1))$ $(\rho\to\infty)$ and $I_\rho^{-1} I_{\nu\rho}(\delta) = A_{0\rho}^{-1}(\delta) A_{\nu\rho}(\delta) \times (1+o(1))$ $(\nu=1,2;\ \rho\to\infty)$. Using (48), this gives after some calculations

$$I_\rho^{-1} I_{1\rho}(\delta) = 1/2 \exp(-\rho \log 2/(\log\rho \log_2^2 \rho)(1+o(1)))$$

and

$$I_\rho^{-1} I_{2\rho}(\delta) = 1/3 \exp(-\rho \log 3/(\log\rho \log_2^2 \rho)(1+o(1))).$$

Moreover, it is not difficult to show that the functions $J_{\nu\rho}$ $(\nu=0,1,2)$ and K_ρ, defined in (7) and (8), are all of $o(I_{2\rho}(\delta))$. Hence, substitution of these results in (9) yields

$$\exp\{\rho\log 2/(\log\rho\log_2^2\rho)(1+o(1))\}.\{(U_\rho f)(x) - f(x)\}$$
$$= -f'(x)(1+o(1))/2 \quad (\rho\to\infty).$$

Thus, if $f'(x) \neq 0$, the order of approximation at x is equal to

$$\exp\{-\rho\log 2/(\log\rho\log_2^2\rho)(1+o(1))\}.$$

If $f'(x)=0$, $f''(x)\neq 0$, the order of approximation at x is higher, viz.

$$\exp\{-\rho\log 3/(\log\rho\log_2^2\rho)(1+o(1))\}.$$

6. Generalization. Let $\delta = e_{n+4}^{-1}$ and $n\in N$ $(n\geqq 4)$. If β is now defined by

$$\beta(0)=1,$$
$$\beta(t) = 1-(\log_{n+1}(1/t))^{-1} \qquad (0<t\leqq\delta),$$
$$(51) \qquad \beta(t) = \alpha e^{-t^2} \qquad (t>\delta),$$
$$\beta(t) = 1-(\log_{n+2}(1/|t|))^{-1} \qquad (-\delta\leqq t<0),$$
$$\beta(t) = \alpha' e^{-t^2} \qquad (t<-\delta),$$

in which α and α' are chosen such that $\beta(t)$ is continuous on R, then it can be proved with the method, used in § 3, that the function

$$\lambda(\rho, u) = \gamma^{-1} u - \rho\log(1-(\log_n u)^{-1}) \quad (0<\gamma\leqq 1,\ \rho\geqq 1,\ u\geqq e_{n+4})$$

has one and only one zero on $[e_{n+4}, \infty)$, if ρ is sufficiently large. This root $u=u_0^*$ gives a minimum for $\lambda(\rho,u)$ and it can be written as

$$u_0^* = \sigma v^*/(\log\sigma\ \log_2\sigma\ \log_3\sigma \ldots \log_{n-1}\sigma \log_n^2\sigma),$$

in which v^* is a power series in the $n+1$ variables

$$\mu_1 = (\log_n\sigma)^{-1},\ \mu_2 = (\log_{n-1}\sigma)^{-1}, \ldots, \mu_n = (\log\sigma)^{-1},$$
$$\mu_{n+1} = (\log_2\sigma + \log_3\sigma + \ldots + \log_n\sigma + 2\log_{n+1}\sigma)/\log\sigma,$$

convergent for all sufficiently small values $\mu_1, \ldots, \mu_{n+1}$.

Furthermore, it can be shown that, if the operators U_ρ are constructed with $\beta(t)$, defined by (51) instead of by (5), the order of approximation at a point x, at which $f''(x)$ exists, is

$$\exp\{-\rho\log 2/(\log\rho\log_2\rho \ldots \log_{n-1}\rho\log_n^2\rho)(1+o(1))\}.$$

7. Conclusion. From the above sections it is clear that for all $n\in N$ the class B contains a β such that the order of approximation of $f(x)$ by $(U_\rho f)(x)$, if $\rho\to\infty$, is equal to $\exp(-\rho/\alpha(\rho))$ for all $f\in M$, for which $f''(x)$

at the point x exists, with $\alpha(\rho)=(\log 2)^{-1}\log\rho\log_2\rho\ldots\log_{n-1}\rho\log_n^2\rho\,(1+o(1))$. Obviously, the larger n, the slower $\alpha(\rho)$ tends to infinity. Thus this result strengthens a conjecture of Totik mentioned in the Introduction.

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