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EXPANSIONS OF DETERMINANT QUOTIENTS WITH APPLICATION TO APPROXIMATION

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Summary. A unified method of dealing with some interpolation and approximation problems, connected with expanding of determinant quotients, is presented. In particular, a determinant extension of de La Vallee-Pousin, Remes and Meinardus-Taylor estimations for the error of the best Chebyshev approximation from below is given.

1. Introduction. Let $A=[a_{ik}]_1^n$ be a non-singular square matrix of order n. Throughout this paper we assume that the matrix $B=[b_{ik}]_1^n$ is defined by

$$b_{ik}=a_{ik},\quad b_{in}=b_i\quad (i=1,\dots,n,\quad k=1,\dots,n-1),$$

where $b_1,\dots,b_n$ are arbitrary scalars. In a few fields of numerical analysis some quantities are defined as determinant quotients of the form $|B|/|A|$. For instance, the error of the best discrete Chebyshev approximation, generalized divided differences and elements of sequences, occurring in the acceleration of convergence of scalar sequences, can be expressed in this way. The purpose of this paper is to present a simple and unified approach to expanding of $|B|/|A|$ in terms of determinant quotients of the same type, but of lower order and to give some applications of the obtained expansions. In particular, a determinant inequality, which can be regarded as an extension of de La Vallee-Pousin [6], Remes [11] and Meinardus, Taylor [7] estimations for the error of the best Chebyshev approximations from below, is given.

2. Main Results. We begin by proving an auxiliary lemma, which provides us with an expansion of a determinant.

Lemma 1. *Let a matrix $A=[a_{ik}]_1^n$ and an integer $m=n-r$ $(1\leq r<n)$ be given. Assume that*

$$A\binom{v+1,\dots,m+v-1}{1,\ \dots,\ m-1}\neq 0\quad (v=0,\dots,r). \tag{1}$$

Then there exist uniquely determined coefficients α_{rv} $(v=0,\dots,r)$ independent of a_{in} $(i=1,\dots,n)$ such that

(2)
$$|A|=\sum_{v=0}^{r}\alpha_{rv}A\begin{pmatrix}v+1, & \dots, & m+v-1, & m+v\\ 1, & \dots, & m-1, & n\end{pmatrix}.$$

P r o o f. Let us consider the following lower-triangular system of $r+1$ linear equations for $r+1$ unknowns α_{rv}

(3)
$$\sum_{v=0}^{r}(-1)^{r-v}\alpha_{rv}A\begin{pmatrix}v+1, \dots, k-1, & k+1, \dots, m+v\\ 1, & \dots, m-1\end{pmatrix}$$
$$=A\begin{pmatrix}1, \dots, k-1, k+1, \dots, n\\ 1, \dots, n-1\end{pmatrix}\quad (k=n, n-1, \dots, m),$$

where it is assumed that the minors on the left side are replaced by zero if $k<v+1$ or $k>m+v$. By (1) all elements of the main diagonal of system' (3) are nonzero, and so (3) has the unique solution α_{rv} $(v=0, \dots, r)$. Now, let us suppose for a moment that A satisfies

(4)
$$a_{in}=0 \quad (i=1, \dots, m-1).$$

Then, multiplying the k-th equation of (3) by $(-1)^{n+k}a_{kn}$ and next summing up the obtained equations, we conclude by the Laplacian determinant expansion formula that formula (2) holds in this case. Otherwise, if A does not satisfy (4), then it can be reduced to a matrix $D=[d_{ik}]_1^n$, satisfying (4) (i. e. such that $d_{in}=0$ for $i=1, \dots, m-1$) by $m-1$ steps of the Gauss elimination method [4], applied to its first $m-1$ columns (possibly, after their permutations). Note, that minors of A, which occur in (1) and (2), are equal to corresponding minors of the matrix $(-1)^{\sigma}D$ (cf. [4, p. 25]), where σ is a number of permutations of the first $m-1$ columns of A in Gauss' elimination. Hence, we can apply (2) to $|D|=(-1)^{\sigma}|A|$ and obtain also formula (2) for A in this general case. □

If $r=n-1$, then the system of equations (3) reduces to a diagonal system and

(5)
$$\alpha_{n-1,v}=(-1)^{n-v-1}A\begin{pmatrix}1, \dots, v, v+2, \dots, n\\ 1, \dots, n-1\end{pmatrix}\quad (v=0, \dots, n-1).$$

Thus, in this case the expansion of $|A|$, given in Lemma 1, coincides with the determinant Laplacian expansion of $|A|$ with respect to the last column. Moreover, by using expansion (2) to the matrices A such that $a_{in}=\delta_{i1}$ (δ_{i1} — Kronecker's delta) and $a_{in}=\delta_{in}$ $(i=1, \dots, n)$, we obtain

(6)
$$\alpha_{r0}=(-1)^{r}\frac{A\begin{pmatrix}2, \dots, n\\ 1, \dots, n-1\end{pmatrix}}{A\begin{pmatrix}2, \dots, m\\ 1, \dots, m-1\end{pmatrix}}$$

and

(7)
$$\alpha_{rr}=\frac{A\begin{pmatrix}1, \dots, n-1\\ 1, \dots, n-1\end{pmatrix}}{A\begin{pmatrix}r+1, \dots, n-1\\ 1, \dots, m-1\end{pmatrix}}.$$

L e m m a 2. *The coefficients* α_{rv} *and* $\alpha_{r+1,v}$ *satisfy the following recurrent formulae:*

$$\alpha_{r+1,v}=\alpha_{r,v-1}\frac{A\begin{pmatrix}v,\dots, & v+m-2\\ 1,\dots, & m-1\end{pmatrix}}{A\begin{pmatrix}v+1,\dots, & v+m-2\\ 1,\dots, & m-2\end{pmatrix}}-\alpha_{rv}\frac{A\begin{pmatrix}v+2,\dots, & v+m\\ 1,\dots, & m-1\end{pmatrix}}{A\begin{pmatrix}v+2,\dots, & v+m-1\\ 1,\dots, & m-2\end{pmatrix}}$$

$$(v=0,\dots,r+1)$$

where $\alpha_{r,-1}=\alpha_{r,r+1}=0.$

Proof. If $D=[d_{ik}]_1^n$ is the adjugate matrix of a matrix $C=[c_{ik}]_1^n$, then by using the Jacobi theorem [1, p. 98], we obtain

$$D\begin{pmatrix}1 & , n\\ n-1, & n\end{pmatrix}=|C|\;C\begin{pmatrix}2,\dots, & n-1\\ 1,\dots, & n-2\end{pmatrix}.$$

In view of the definition of the adjugate matrix it follows

$$|C|\;C\begin{pmatrix}2,\dots, & n-1\\ 1,\dots, & n-2\end{pmatrix}=C\begin{pmatrix}2,\dots, & n-1, & n\\ 1,\dots, & n-2, & n\end{pmatrix}C\begin{pmatrix}1,\dots, & n-1\\ 1,\dots, & n-1\end{pmatrix}$$

$$-C\begin{pmatrix}1,\dots, & n-2, & n-1\\ 1,\dots, & n-2, & n\end{pmatrix}C\begin{pmatrix}2,\dots, & n\\ 1,\dots, & n-1\end{pmatrix}.$$

Hence

$$A\begin{pmatrix}v+1,\dots, & & m+v\\ 1,\dots, & m-1, & n\end{pmatrix}$$

$$=\frac{A\begin{pmatrix}v+2,\dots, & m+v & \\ 1,\dots, & m-2, & n\end{pmatrix}A\begin{pmatrix}v+1,\dots, & m+v-1\\ 1,\dots, & m-1\end{pmatrix}-A\begin{pmatrix}v+1,\dots, & m+v-1 & \\ 1,\dots, & m-2, & n\end{pmatrix}A\begin{pmatrix}v+2,\dots, & m+v\\ 1,\dots, & m-1\end{pmatrix}}{A\begin{pmatrix}v+2,\dots, & m+v-1\\ 1,\dots, & m-2\end{pmatrix}}$$

Finally, setting this expression in (2) and arranging the obtained sum with respect to $A\begin{pmatrix}v+1,\dots, & m+v-2, & m+v-1\\ 1,\dots, & m-2, & n\end{pmatrix}$, we derive the required recurrent formulae. □

Now, let us suppose that the matrix A is non-singular. Then the following corollary follows easily from the Lemma 1.

Corollary 1. *Let* $m=n-r$ $(1\leq r<n)$ *and*

$$A\begin{pmatrix}v+1,\dots, & m+v-1\\ 1,\dots, & m-1\end{pmatrix}A\begin{pmatrix}v+1,\dots, & m+v-1, & m+v\\ 1,\dots, & m-1, & n\end{pmatrix}\neq 0\quad (v=0,\dots,r).$$

Then

$$\frac{|B|}{|A|}=\sum_{v=0}^{r}\lambda_{rv}\frac{B\begin{pmatrix}v+1,\dots, & m+v-1, & m+v\\ 1,\dots, & m-1, & n\end{pmatrix}}{A\begin{pmatrix}v+1,\dots, & m+v-1, & m+v\\ 1,\dots, & m-1, & n\end{pmatrix}},\tag{8}$$

where coefficients λ_{rv} *are independent of* $b_1,\dots,b_n$ *and equal to* $\lambda_{rv}=\alpha_{rv}A\begin{pmatrix}v+1,\dots, & m+v-1, & m+v\\ 1,\dots, & m-1, & n\end{pmatrix}/|A|$. *Moreover*

$$\sum_{v=0}^{r}\lambda_{rv}=1.\tag{9}$$

P r o o f. By applying Lemma 1 to the matrix B we get the desired expansion for $|B|/|A|$. Formula (9) follows immediately from (8) after setting $b_i = a_{in}$ $(i=1, \dots, n)$ in (8). □

In the next theorem we establish an inequality for determinant quotients. We shall see in the following that this inequality can be regarded a sa determinant generalization of de La V a l l e e-P o u s i n [6, p. 82], R e m e s [11, p. 40] and M e i n a r d u s, T a y l o r [7] estimations from below for the error of the best Chebyshev approximation. For this purpose denote

$$T_{rv} = \frac{B\begin{pmatrix} v+1, \dots, & m+v-1, & m+v \\ 1\ , \dots, & m-1, & n \end{pmatrix}}{A\begin{pmatrix} v+1, \dots, & m+v-1, & m+v \\ 1\ , \dots, & m-1, & n \end{pmatrix}}.$$

T h e o r e m 1. *Under the assumptions of Corollary 1 and the additional assumption that there exist* $\sigma_1, \sigma_2 \in \{-1, 1\}$ *such that* $\operatorname{sign}(\lambda_{rv}) = \sigma_1$ *and* $\operatorname{sign}(T_{rv}) = \sigma_2$ $(v=0, \dots, r)$, *we have* $||B|/|A|| \geq \min\{|T_{rv}| : 0 \leq v \leq r\}$.

P r o o f. From Corollary 1 and the assumptions of the theorem it follows that $||B|/|A|| = |\Sigma_{v=0}^{r} \lambda_{rv} T_{rv}| = \Sigma_{v=0}^{r} |\lambda_{rv}|\,|T_{rv}| \geq \Sigma_{v=0}^{r} |\lambda_{rv}| \min\{|T_{rv}| : 0 \leq v \leq r\} = |\Sigma_{v=0}^{r} \lambda_{rv}| \min\{|T_{rv}| : 0 \leq v \leq r\}$. Hence by (9) the proof is completed. □

We note, that from Lemma 1 we obtain immediately the following determinant extension of expansions for generalized divided differences [8, 9, 10] of order n by divided differences of order m and for elements of sequences, occurring in the process of acceleration of convergence of scalar sequences [2, 5].

C o r o l l a r y 2. *Let* $m = n - r$ $(1 \leq r < n)$ *and* $A\begin{pmatrix} v+1, \dots, & m+v-1 \\ 1\ , \dots, & m-1 \end{pmatrix} \times A\begin{pmatrix} v+1, \dots, & m+v \\ 1\ , \dots, & m \end{pmatrix} \neq 0$ $(v=0, \dots, r)$. *Then*

$$|B|/|A| = \sum_{v=0}^{r} \beta_{rv} B\begin{pmatrix} v+1, \dots, & m+v-1, & m+v \\ 1\ , \dots, & m-1, & n \end{pmatrix} / A\begin{pmatrix} v+1, \dots, & m+v \\ 1\ , \dots, & m \end{pmatrix}, \tag{10}$$

where coefficients β_{rv} *are independent of* $b_1, \dots, b_n$ *and*

$$\beta_{rv} = \alpha_{rv} A\begin{pmatrix} v+1, \dots, & m+v \\ 1\ , \dots, & m \end{pmatrix} / |A|.$$

Moreover

$$\sum_{v=0}^{r} \beta_{rv} = 0. \tag{11}$$

P r o o f. Analogously as in Corollary 1, we must only prove (11). But this equality follows directly from formula (10) after setting $b_i = a_{mi}$ $(i=1, \dots, n)$ there. □

3. Applications. Now, we discuss briefly some earlier mentioned applications of results, obtained in the preceding section. Suppose that $G_i = \operatorname{span}\langle g_1, \dots, g_i \rangle$ $(i=1, \dots, n-1)$ are given i-dimensional subspaces of the space $C(I)$, $I=[a, b]$, of real-valued continuous functions, defined on I, and that G_{n-1} is $n-1$ dimensional Haar space on I, i. e. that functions $g_1, \dots, g_{n-1}$ form the Chebyshev system on I. If the sets $X = \{x_i\}_1^n$ $(a \leq x_1$

$<\cdots<x_n\leq b)$ and Y $(X\subset Y\subset I)$, $f\in C(I)$ and $g\in G_{n-1}$ are chosen arbitrarily, then by the Alternation Theorem [3, p. 75] we have

$$(12)\qquad \rho(f,\ Y)\geq\rho(f,\ X)=\rho(e,\ X)=||B|/|A||,$$

where $\rho(h,\ Z)$ denotes the error of the best Chebyshev approximation of h by elements of G_{n-1} on a subset Z of I, $e=f-g$ and matrices $A=[a_{ik}]_1^n$ and $B=[b_{ik}]_1^n$ of the same type as in Section 2 are defined by

$$(13)\qquad a_{ik}=b_{ik}=g_k(x_i),\ a_{in}=(-1)^i \text{ and } b_{in}=e(x_i)$$
$$(i=1,\ldots,n \text{ and } k=1,\ldots,\ n-1).$$

Since $\operatorname{sign}|A|=(-1)^n$ and determinants in formula (5) for $\alpha_{n-1,v}$ are positive, we conclude from Corollary 1 that $\operatorname{sign}(\lambda_{n-1,v})=1$ and $T_{rv}=(-1)^{v+1}\times e(x_{v+1})$ $(v=0,\ldots,\ n-1)$. Hence the assumptions

$$(14)\qquad e(x_v)e(x_{v+1})<0\quad (v=0,\ldots,n-1)$$

are sufficient for the existence of σ_2, defined in Theorem 1. By (12) the determinant inequality, given in this theorem for $r=n-1$, provides us with the well-known de La Vallee Pousin estimation from below for the error $\rho(f,\ Y)$. Similarly, if additionally $G_{n-2},\ldots,G_{m-1}$ $(m=n-r,\ 1\leq r<n-1)$ are Haar spaces on I, then using (6), (7) and Lemma 2 we obtain by an easy induction with respect to r that $\operatorname{sign}\alpha_{rv}=(-1)^{r-v}$. Since

$$\operatorname{sign}A\begin{pmatrix}v+1,\ldots, & m+v-1, & m+v\\ 1,\ldots, & m-1, & n\end{pmatrix}=(-1)^{m+v},$$

then by Corollary 1 we have $\operatorname{sign}(\lambda_{rv})=1$. Moreover, there exists σ_2, defined in Theorem 1 if and only if the signs of elements in the following sequence of minors

$$B\begin{pmatrix}v+1,\ldots, & m+v-1, & m+v\\ 1,\ \ldots, & m-1, & n\end{pmatrix}\qquad (v=0,\ldots,r)$$

oscillate. If we apply the Laplacian expansion formula to these minors with respect to their last columns, then we deduce that conditions (14) are sufficient for the existence of σ_2 from Theorem 1. In this case the estimations of the error $\rho(f,\ Y)$ from below are reduced to the estimations of Remes [11] and Meinardus, Taylor [7]. Since $T_{rv}=\rho(f,\ X_v)$, where $X_v=\{x_{v+1},\ldots,x_{m+v}\}$, then from Theorem 1 it follows that the estimation of the error $\rho(f,\ Y)$ for r_1 is not worse than the estimation for r_2, while $r_1<r_2$. In particular, if G_{n-1} is equal to the set of all polynomials of degree less than or equal to $n-2$, then under the assumption (14) we have

$$\rho(f,\ Y)\geq\frac{1}{2}\min\{(\lambda_v|e_v|+|e_{v+1}|+(1-\lambda_v)|e_{v+2}|):\ 1\leq v\leq n-2\}$$
$$\geq\frac{1}{2}\min\{(|e_v|+|e_{v+1}|):1\leq v\leq n-1\}\geq\min\{|e_v|:\ 1\leq v\leq n\},$$

where $\lambda_v=(x_{v+2}-x_{v+1})/(x_{v+2}-x_v)$ and $e_v=e(x_v)$.

We note that we are going in a forthcoming paper to obtain analogous estimations of the error of the best Chebyshev approximation [12] with Hermite constrains. By Theorem 1 this problem requires only determining of signs of the minors occurring in λ_{rv} and T_{rv}. Finally, if $G_n = \text{span}\langle g_1, \ldots . g_n\rangle$ is the Haar space on X and the elements of matrices A and B are defined as in (13), except of elements a_{in}, equal to $g_n(x_i)$ $(i=1, \ldots, n)$, then determinant quotient $|B|/|A|$ is the generalized divided difference [8] at points x_i and with respect to G_n. Hence we can apply Corollary 2 and obtain expansion of them according to generalized divided differences of a lower order (cf., e. g. [8, 9, 10]).

REFERENCES

1. A. C. Aitken. Determinants and Matrices. 9th ed. Edinburgh, 1956.
2. C. Brezinski. Accélération de la Convergence en Analyse Numèrique. (Lect. Notes Math., **584**). Berlin, 1977.
3. E. W. Cheney. Introduction to Approximation Theory. New York, 1966.
4. F. R. Gantmacher. The Theory of Matrices. New York, 1964.
5. T. Håvie. Generalized Neville type extrapolation schemes. *BIT*, **19**, 1979, 204-213.
6. G. Meinardus. Approximation of Functions: Theory and Numerical Methods. Berlin, 1967.
7. G. Meinardus, G. D. Taylor. Lower estimates for the error of best uniform approximation. *J. Approx. Theory*, **16**, 1976, 150-161.
8. G. Mülbach. A recurrence formula for generalized divided differences and some applications. *J. Approx. Theory*, **9**, 1973, 165-172.
9. G. Mülbach. The general Neville-Aitken-algorithm and some applications. *Numer. Math.*, **31**, 1978, 97-110.
10. G. Mülbach. The general recurrence relation for divided differences and the general Newton-interpolation-algorithm with applications to trigonometric interpolation. *Numer. Math.*, **32**, 1979, 393-408.
11. E. Ya. Remes. General Computational Methods of Tchebycheff Approximation. Kiev, 1969.
12. G. D. Taylor, H. L. Loeb, D. G. Moursund, L. L. Schumaker. Uniform generalized weight function polynomial approximation with interpolation. *SIAM J. Numer. Anal.*, **6**, 1969, 284-293.

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