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# OPTIMAL PASSIVE AND SEQUENTIAL ALGORITHMS FOR THE APPROXIMATION OF CONVEX FUNCTIONS IN $L_p([0, 1]^s)$, $p=1, \infty$

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***Summary.*** We present optimal, resp. asymptotically, ($N\to\infty$) optimal algorithms for the passive (simultaneous) and sequential choices of nodes for the $L_1$ and $L_\infty$ approximation of convex functions over $[0, 1]^s$, $s\geqq 1$. In contrast to problems of individual best approximation, here the criterion is the global error of approximation over some naturally chosen class of functions (e. g. specified by their, fixed values on the boundary). An optimal, nonlinear quadrature (cubature) formula is obtained, with nearly Chebyshev (with respect to their distribution of) nodes. Sequential $N$-step algorithms for choosing the nodes are described, which yield a more satisfactory solution of the cubature problem, and which in the case of $L_\infty$, i. e. uniform, approximation, have global errors of order (in $N$) significantly less than the optimal passive $N$-step algorithms. The construction principles and the corresponding error estimations, given here, yield, in most cases, asymptotically optimal algorithms, whose global errors have the same order as the maximal order of individual best $N$ point approximations within the class.

**2. Definitions. Preliminaries.** First we define the notions of optimal passive, resp. sequential $N$-step algorithms for approximating — in the $L_1$ or $L_\infty$ sense — the elements of a class of functions

$$K^0=\{x: x \text{ convex in } t\in T,\ x(t_i^0)=c_i,\ t_i^0\in T,\ i\in I_0\}, \tag{1}$$

where $T$ is a subset of $R^p$ — in the following $T$ will always be an 'interval" i. e. the direct product of $p$ one-dimensional intervals, $I_0$ be a finite or infinite index set, $|I_0|\geqq(p+2)^p$, more precisely we consider two 'standard' cases, assuming that the values of $x$ are fixed, at least in one case, $W_1$: at the boundary of $T$, together with the (sub)gradients (see below (6), (10)); and in the other case, $W_2$: at the centres of all $q$-dimensional faces of $T$, $q=0, 1, \ldots, p$.

A sequential $N$-step algorithm is given by $N$ functions $A^N=(A_1, \ldots, A_N)$, defined over the set of (finite codimensional) linear '$t$-sections' of $K^0$

$$S(K^0, T)=\{s(t^k, c^k) \mid t^k\in T^k, c^k\in R^k,\quad k=0, 1, 2, \ldots\},$$

where $t^k=(t_1, \ldots, t_k)$ belongs to a specified subset of $T$ (see below) and $c^k=(c_1, \ldots, c_k)$,

$$s(t^k, c^k):=\{x \mid x\in K^0,\ x(t_i)=c_i,\ i=1, \ldots, k\},$$

$$t_{n+1}:=A_{N-n}(s(t^n, x)),\ \text{where}\ \begin{vmatrix} s(t^n, x):=s(t^n, c^n), \\ x(t_i)=c_i,\ i=1, \ldots, n, \end{vmatrix} \tag{2}$$

$n=0, 1, \ldots, N-1$, thus $A_n: S(K^0, T) \to T$ we could allow $A_n$ to be multivalued, yet for simplicity we do not. To each $A^N$ and $x$ corresponds a $t^N = t^N(A^N, x)$ and

$$s(A^N, x) := s(t^N(A^N, x), x).$$

The global error of $A^N$ over $K^0$ is defined as

$$e(A^N, K^0) := \sup \{E^d(s(A^N, x)) : x \in K^0\}, \tag{3}$$

where $E^d(L)$ is the diameter of the set $L$, in the space $L_1(T)$ or $L_\infty(T)$ When needed, we shall distinguish the case in question by a lower index $p$, $p=1$, or $p=\infty$.

Remark. It is a special feature of our approximation problems, chiefly of the set $K^0$, that in each section $s(t^N, c^N)$ there exists a maximal element (function) $x^M$ with respect to the natural order: $x_1 \geqq x_2$, if $x_1(t) \geqq x_2(t)$, for all $t \in T$. $x^M$ is given by the convex hull of the points $(t_i, x(t_i))$, $i=1, \ldots, N$, $i \in I_0$, in an obvious way, thus computing the diameter is just computing the maximal deviation — within $s(t^N, c^N)$ — from $x^M$ in the norm considered; the $L_1$ problem is thus equivalent to a quadrature, cubature, ... problem.

An $N$-step algorithm $\bar{A}^N$ is said to be optimal, if

$$e(\bar{A}^N, K^0) = \inf_{A^N} e(A^N, K^0) =: e_s(N, K^0). \tag{4}$$

The optimal passive 'open loop' $N$-step algorithms, $\bar{t}^N$ are defined — if they exist — by the requirement

$$e(\bar{t}^N, K^0) = \sup_{x \in K^0} E^d(s(t^N, x)) = \inf_{t^N} \sup_{x \in K^0} E^d(s(t^N, x)) =: e_p(N, K^0), \tag{5}$$

here $t^N$ belongs to a specified subset, $T_i^N$, of the direct product $T^N$; to be more precise we shall later distinguish two cases, $i=0, 1$:

$M_0$: each measurement point $t_j$ in $t^N$ has multiplicity one;

$M_1$: each measurement point $t_j$ in $t^N$ has multiplicity two,

i. e. in the case $M_1$ at each measurement, step $n$, both $c=x(t)$ and $g=\operatorname{grad} x(t)$ are measured (computed) at a point $t=t_{n+1}$; for simplicity of notation even in the case $M_1$ we shall speak about $N$-step algorithms and write $t^k$ instead of $t^{2k}$, and $c^k$ instead of $(c^k, g^k)$ and $T^N$ instead of $T_i^N$, $i=0, 1$.

In fact we shall consider the case $M_1$ only for the one-dimensional problems, when the measurement at $t$ of $c$ and $g$ gives the information, that

$$x(s) \geqq c + g(s-t), \text{ for all } s, \text{ i. e. } g \text{ is a subgradient.} \tag{6}$$

Proposition 1. Suppose that the function has a saddle point on $T^N \times K^0$, $(\bar{t}^N, \bar{x})$,

$$e(t^N, \bar{x}) \geqq e(\bar{t}^N, \bar{x}) \geqq e(\bar{t}^N, x) \quad \text{for all} \quad (t^N, x) \in T^N \times K^0, \tag{7}$$

then we have equality for the global errors of optimal passive and sequential algorithms:

$$e_p(N, K^0) = e_s(N, K^0). \tag{8}$$

The proof is simple. Firstly one shows that (7) implies that $e_p(N, K^0)=e(\bar{t}^N, \bar{x})$, then one observes that for any $A^N$

$$e(A^N, K^0)\geq E^d(s(t^N(A^N, \bar{x}), \bar{x}))\geq e(\bar{t}^N, \bar{x}),$$

by the left-hand side of (7), q. e. d.; see [1, Ch. V, § 8] for a special case of the validity of (7).

Of course passive algorithms are much more easily realizable, however, for the sets $K^0$ (8) holds only in exceptional cases. First we describe these for the $L_1$ problem.

Optimal, in the sense (4), sequential algorithms are — if (8) does not hold— in general very complicated. On the other hand, one can often find relatively easily realizable infinitely continuable $N$ step algorithms, $B^\infty$ $t_{n+1}:=B_{n+1}(s(t^n, x))$, thus here we have no dependence on time horizons $N$, which are asymptotically optimal: for some constant $a(K^0)$,

$$e(B^N, K^0)=\sup_{x\in K^0} E^d(s(t^N(B^\infty, x), x))\leq a(K^0)e_s(N, K^0)$$

holds for all $N$.

In the case of the $L_\infty$, i. e. uniform approximation problem — where, in order that $e(A^N, K^0)\to 0$ for $N\to\infty$ be possible, we have to assume that the gradients are given (or at least bounded) at the boundary of $T$, — it is easy to show that, if in (1) there are only a finite number of points $t$ with fixed $x$ values, then for some constant, $f_1(K^0)$, $e(t^N, K^0)\geq f_1(K^0)N^{-1/p}$ for all $t^N$.

By sequential choices of nodes one can achieve global errors of order, in $N$, less than $1/p$, perhaps, seé below Section 4,

$$e(A^N, K^0)\leq f_2(K^0)N^{-2/p}, \tag{9}$$

which is the best possible order: a function $x$ in $K^0$, which is quadratic with positive definite Hesse matrix, $D^2x$, on some open domain of $T$ cannot be better reconstructed — in both norms $L_1, L_\infty$ — from $N$ measurements.

It is easy to show that in the case of the metrics $L_1$ and of a priori bounded (given) gradients — but not for the case $W_2$, see below (16), already the uniform, equidistanced set of nodes yields the optimal order $-2/p$.

In order to find (construct) good $N$-step algorithms, one should study the characterization of the nodes $\bar{t}^N(x)$ of the individually best $N$ point approximations, i. e. thosé realizing $\inf_{t^N} E^d(s(t^N, x)) =: e_p(N, x)$ $(p=1, \infty)$.

In the one-dimensional case the asymptotics

$$e_1(N, x)=8^{-1}N^{-2}\left(\int_a^b \sqrt[3]{x''(s)}ds\right)^3+\sigma(N^{-2}),\quad e_\infty(N, x)=4^{-1}N^{-2}\left(\int_a^b\sqrt{x''(s)}\,ds\right)^2$$

$$+\sigma(N^{-2}),$$

are easily proved (e. g. for $x''\geq d_0>0$, and continuous) and the principle: the 'local errors', corresponding to the node free subintervals, must be equal, see e. g. [1, Ch. III, § 9], justified. An interesting problem is whether there exists an infinitely continuable algorithm $B^\infty$ and a constant, $k_p(K^0)$, such that $e(B^N, x)\leq k_p(K^0)e_p(N, x)+\sigma(e_s(N, K^0)$, see below (20)-(22).

**3. Optimal Passive $N$-step Algorithms of Integration.** When not stated otherwise, in this section we deal with the $L_1$-approximation problem.

It turns out that the multidimensional case $p>1$ for 'intervals' $T=[a_1, b_1]\times\ldots\times[a_p, b_p]$ is easily reducible to the one-dimensional case.

We say that $K^0(a, b, c_0, c_\infty, d_0, d_\infty)=K^0$ is a $W_1$ standard set (or simply: a standard one in the one-dimensional case), if

$$K^0=\{x: x \text{ convex}, x(a)=c_0, x(b)=c_\infty, x'(a)=d_0, x'(b)=d_\infty\}, \tag{10}$$

thus, with multiplicities, here $|I_0|=4$. The importance of these types of sets $K^0$ is that by fixing the results of $n$ additional measurements of $x$ and grad $x$ at $t_1\leqq\ldots\leqq t_n\in(a, b)$, the behaviour of the function $x$ is decomposed into $n+1$ independent parts, each of them of the same structure as $K^0$. In fact a simple, linear, direct sum decomposition arises

$$K^0(a, b)=\sum_i K^0(t_i, t_{i+1}),\ t_0=a,\ t_{n+1}=b,\ 0\leqq i\leqq n, \tag{11}$$

where we use the abbreviation $K^0(t_i, t_{i+1})=K^0(t_i, t_{i+1}, x(t_i), x(t_{i+1}), x'(t_i), x'(t_{i+1}))$ and one immediately sees, that

$$\begin{aligned} E_1^d(K^0)&=\sum_i E_1^d(K^0(t_i, t_{i+1})),\\ E_\infty^d(K^0)&=\max_i E_\infty^d(K^0(t_i, t_{i+1})).\end{aligned} \tag{12}$$

In order to describe the optimal set of $N$ nodes for the standard set $K^0$, let us denote by $ABC$ the smallest triangle, $A=(a, c_0)$, $B=(b, c_\infty)$, $C=(\bar{s}, y)$, which contains the graphs of functions in $K^0$. Let

$$\bar{t}_{N+1-k}:=k(ka+(N+1-k)\bar{s})+(N+1-k)(k\bar{s}+(N+1-k)b),\ k=1,\ldots, N, \tag{13}$$

$$\bar{s}=(c_\infty-c_0+ad_0-bd_\infty)/(d_0-d_\infty),\quad y=c_0+d_0(\bar{s}-a).$$

Theorem 1. *For the standard set $K^0$ (10) for both cases of the information pattern $M_0$ and $M_1$, the set of nodes $\bar{t}^N$, given by* (13), *and $\bar{x}$, the parabola, which is tangential to the sides $AC$ and $BC$ at $A$ and $B$, constitute a saddle point* (7) *for the integration problem, thus for the latter* (8) *holds for all $N$*

$$\begin{aligned} e_s(N, K^0)&=e_p(N, K^0)=E_1^d(K^0)(N+1)^{-2},\\ E_1^d(K^0)&=2^{-1}(b-a)(c_0+(c_\infty-c_0)(b-a)^{-1}(\bar{s}-a)-s).\end{aligned} \tag{14}$$

Proof. First note that the axe of the parabola $\bar{x}$ is parallel to the segment $[C, (A+B)/2]$ and the points $t_k$ are abscissae of the points $P_k$, which are intersections of $\bar{x}$ with lines, parallel to the axe and going through $kB+(N+1-k)A$. To prove the second inequality in (7) we need a well-known lemma (Archimedes).

Lemma 1. *The area of a triangle, formed by two tangents to the parabola at points $P$, $Q$ and by the segment $[P, Q]$, is proportional to the third power of the length of the projection of $[P, Q]$ — parallel to the axe — into any line, e. g. into that determined by $A$ and $B$.*

Now to prove the statement first for the case $M_1$ we need only the inequality

$$N^2\sum_i m_i^3\geqq(\Sigma m_i)^3. \tag{15}$$

To prove the first (left) inequality in (7) we note that for a fixed $t^N = \bar{t}^N$, $E(s(\bar{t}^N, c^N)$ is a concave function of $c^N$, and the latter expression is a two times continuously differentiable function of $c^N$ near the values $\bar{c}_k = \bar{x}(\bar{t}_k)$, $k = 1, \ldots, N$, moreover, its partial first derivatives with respect to $c_i$ vanish at $\bar{c}^N$ for all $i = 1, \ldots, N$.

The study of the first partial derivatives by the special configuration corresponds just to the case $N=1$ of the problem, for which one can prove the optimality of $\bar{t}_1$, see (13) for $N=1$, $\bar{t}_1 = \bar{s}/2 + (a+b)/4$, easily, omitting simple geometric, algebraic arguments (e. g. that for given $t_1, c_1$, near $(\bar{t}_1, \bar{c}_1)$, the worst value of $x'(t_1)$ is such that for the new sets $K_1^0$, $K_2^0$, $B_1 \equiv A_2$, $\| C_1 - B_1 \| = \| A_2 - C_2 \|$) by noticing that $E^d(s(t_1, c))$, near $\bar{t}_1$, $\bar{c} = \bar{x}(\bar{t}_1)$ is a quadratic function of $(t_1, c)$, for which this point is an extremum, in fact, saddle point.

The existence in both cases of information patterns $M_0$, $M_1$ of the maximal element in a section $s(t^n, c^n)$ and the existence of subderivatives at any points, gives that a saddle point for the case $M_1$ is also a saddle point for the case $M_0$. Finally (14) follows from (15), $E_1^d(K^0) =$ area $ABC$ and from a simple geometric computation using induction with respect to $N$.

Remark 2. By the fact that the parabola and the construction of points $P_k$ are affinely invariant, we obtain that the statement of the theorem remains true also in that case, when $d_0$ or $d_\infty$ (but only one of them) is infinity, i. e. when convex functions are allowed, which might be not even continuous at one of the ends of $[a, b]$. This will now be used.

Definition. We say that $K_2^0 = K^0(a, b, c_0, c_1, c_\infty)$ is a $W_2$ standard set in the one-dimensional case, if

$$K_2^0 = \{x | x \text{ convex}, \quad x(a) = c_0, \; x((a+b)/2) = c_1, \quad x(b) = c_\infty\}. \tag{16}$$

Corollary 1. *For arbitrary $N = 2k$ the information pattern $M_0$ and a $W_2$ standard set $K_2^0$ in* (16)

$$e_s(N, K_2^0) = e(N, K_2^0) = E_1^d(K_2^0)((N/2)+1)^{-2}, \; E_1^d(K_2^0) = 2^{-1}(b-a)(2^{-1}(c_0 + c_\infty) - c_1). \tag{17}$$

The proof follows from the observation that for an arbitrary value of $x'((a+b)/2)$ the sum of the (equal) $L_1$ diameters of the two $W_1$ standard sets with infinite derivatives at one of their ends, which arise from the given $W_2$ standard set, is just the $L_1$ diameter of the latter.

It is interesting to note that the optimal system of $N$ nodes for a $W_2$ standard set is independent of $c_0$, $c_1$, $c_\infty$ and very close to the well-known Chebyshev system of $N$ nodes.

We define the $W$ standard sets $K_{p+1}^0 = K(a_i, b_i, c_{0i}, c_{1i}, c_{\infty i}, i = 1, \ldots, p)$ in the multidimensional case, $p$ arbitrary, by fixing in (1) the values of $x$ at all centres of $q$-dimensional faces of $T$, $q = 0, 1, \ldots, p$.

Corollary 2. *For an arbitrary $W_2$ standard set $K_{p+1}^0$, $T = [-1, 1]^p$, and $N = (2k)^p$, $k$ arbitrary, the passive choice of $N$ nodes, $\bar{t}^N$ given by*

$$\bar{t}_{n,r} = -1 + n^2(k+1)^{-2}, \; \bar{t}_{m,r} = 1 - m^2(k+1)^{-2}, \; n, m = 1, \ldots, k, \; r = 1, \ldots, p$$

*allows us to achieve for some constant $k_0(K_{p+1}^0)$*

(18) $$e_s(N, K^0) \leq e(\bar{t}^N, K^0) \leq k_0(K^0_{p+1})N^{-2/p},$$

*even if we can measure the values of* $x(t)$ *only within errors not greater than* $k_1(K^0_{p+1})N^{-2/p}$.

To prove this we note that the $p-q$ dimensional integrals of $x$ along linear sections of $T$, $T(s_1, \ldots, s_q)$, orthogonal to the first $q$ axes, are convex functions of the variables $s_1, \ldots, s_q$. Thus, (18) is obtained by induction with respect to $p$, noticing that, if in Corollary 1 the measurements of $x(t)$ are assumed to contain errors of magnitude less than $d$, then for some constant $k_2(K^0_2)$

$$e_p(N, K^0_2) \leq (1+N/2)^{-2}(E^d_1(K^0_2) + dk_2(K^0_2)) + d(b-a).$$

Therefore we have only to see that $d$ will be, at all steps of the induction, of order $k^{-2}$, i. e. of $N^{-2/p}$. We note that the optimal quadrature formula for the class $K^0_{p+1}$, $p \geq 1$, is nonlinear, yet its error is only twice less than that of a corresponding linear formula (i. e. giving the integral of the maximal function $x^M(s(\bar{t}^N, c^N))$).

**4. Sequential Methods of Approximation.** First we study the one-dimensional case. If the set $K^0$ (1) is not one of the standard sets considered in Section 3, then even in the $L_1$ problem (which we study now first, making only some remarks concerning the $L_\infty$ approximation problem) sequential algorithms might be globally more accurate in the case of the information pattern $M_0$. A simple example is the following: $|I_0| = 3$, $x(0) = 1$, $x(1/3) = 0$, $x(1) = 1$; $5/72 \geq e_s(2, K^0)$, $e_p(2, K) = 1/12$. By Theorem 1 for all $K^0$

(19) $$4e_s(N, K^0) \geq e_p(N, K^0).$$

We give now a sequential algorithm for the case $M_0$ and for the case when the values of $x$ are computed only within accuracy $\varepsilon_1$.

Algorithm 1. Fix a positive number $\varepsilon_0$. Let $K$ be a $W_1$ standard set, compute the values of $x$ at

$$t_j = a + (b-a)2^{-j}, \quad j = 1, 2, \ldots, p+1,$$

till on the interval $[a, t_p]$ the uncertainty in the value of the integral over this interval — as computed approximately (see below) from the convexity and the computed values of $x$ — is not greater than $\varepsilon_0$. Defining the algorithm by induction, suppose that at step $n$ the interval $[t_{n_k}, t_{n_{k+2}}]$ is the first one from the left, over which the uncertainty in the value of the integral is greater than $\varepsilon_0 + 5\varepsilon_1(t_p - a)/4$; we need only the values of at $t$, $t_{n_k}$, $t_{n_{k+1}}$, $t_{n_{k+2}}$ to compute this uncertainty, while only in the case, when $\varepsilon_1 = 0$ i. e. the measurements are exact, is the exact uncertainty computable from five values. Let $t_{n+1} := (t_{n_k} + t_{n_{k+1}})/2$ be the next measurement and continue in the same way: $E^d_1(s(t^n, c^n), [t_{n_k}, t_{n_{k+1}}]) \leq \varepsilon_0 + 5\varepsilon_1(t_{n_{k+1}} - t_{n_k})/4$.

Theorem 2. *The number of steps,* $N(\varepsilon_0, K^0, x)$, *necessary to end (i. e. reach b in) algorithm A, and thus compute x within accuracy, in* $L_1 - N(\varepsilon_0, K^0, x)\varepsilon_0 + \varepsilon_1(b-a)5/4$ *is less — for all x in* $K^0$ *— than*

$$N_0(K^0, \varepsilon_0) = 2N_1(\varepsilon_0, K^0) - 1, \quad N_1(\varepsilon_0, K^0) := [((x'(b) - x'(a))(b-a)^2/\varepsilon_0)^{1/3} - 1]_+.$$

Proof. In order to prove Theorem 1 we need to introduce an auxiliary algorithm 2, which works as Algorithm 1, but uses the information pattern $M_1$

(exact computation of $x(t_n)$ and $x'(t_n)$ at step $n$). Using them, we have then, see (11), a decomposition and instead of the easily computable exact uncertainty $E^d(K^0_{n_k})$ we could, as will be clear from the proof below, use also the criterion (say for $\varepsilon_1=0$)

$$(20) \qquad (x'(t_{n_{k+1}})-x'(t_{n_k}))(t_{n_{k+1}}-t_{n_k})^2\leq\varepsilon_0.$$

(We remark that $(\int_a^b\sqrt[3]{x''(s)}\,ds)^3\leq(x'(b)-x'(a))(b-a)^2$). The number of steps sufficient to reach — for all $x\in K^0$ — $b$ from $a$ in this algorithm is just $N_0(\varepsilon_0, K^0)$. This can be proved by induction with respect to the 'length' $(b-a)$ of $K^0$, based on the following Lemma.

Lemma 2. *For a $W_1$ standard set $K^0$ one has*

$$(21) \qquad E_1^d|K^0|\leq(x'(b)-x'(a))(b-a)^2/8=:D_1(K^0).$$

*This follows from a simple similar inequality*

$$(22) \qquad E_\infty^d(K^0)\leq(x'(b)-x'(a))(b-a)/4=:D_\infty(K^0).$$

The first step of the induction is established now: $N_1(\varepsilon_0, K^0)<1$ implies by (21) $E_1^d(K^0)\leq\varepsilon_0$, thus no more measurements are needed. The induction step is based on the inequality $1+N_1(\varepsilon_0, K_1^0)+N_1(\varepsilon_0, K_2^0)\leq N_1(\varepsilon_0, K^0)$, i. e.

$$x^{1/3}+y^{1/3}\leq 4(x+y)^{1/3} \quad \text{for} \quad x, y>0;$$

here the monotonity of the gradient $x'$ is used.

To finish the proof of Theorem 1 we need only:

Lemma 3. *On each measurement free subinterval, arising from th( application of Algorithm 2, there is at most one additional point, arisine in Algorithm 1, i. e. if for a $W_1$ standard set $K^0$ the values of $x$ are computed within accuracy $\varepsilon_1$ at the points $a$, $(a+b)/2$, $b$, then the uncertainty in the value of the integral over $[a, b]$ is not greater than $2(E^d(K^0)+\varepsilon_1(b-a)5/8)$, for all $W_1$ standard sets, $K^0$, compatible with these three measurements.*

In order to get an infinitely continuable algorithm $B^\infty$, let us repeat Algorithm 1 with $\varepsilon_0=(1/8)^k$, $\varepsilon_1=\varepsilon_0^{2/3}$, $k=1, 2, \ldots$ (at 'megastep' $k$ one could forget all the previous measurements, yet the above sequence of choices of $\varepsilon_0$ is made in order that one could maximally use the information, got during the previous megasteps). This is easily proved, when Theorem 2 is stated in the equivalent form: for $N=N_0(\varepsilon_0, K^0)$ and Algorithm 1

$$e(A^N, K^0)\leq 8(x'(b)-x'(a))(b-a)^2(1+N)^{-2}+5\varepsilon_1(b-a)/4,$$

especially $\varepsilon_0\leq N^{-3}(\varepsilon_0, K^0)\,c_0$, for some constant $c_0(K^0)$.

Let us note that for the $L_\infty$ problem the analogues of Algorithms 1 and 2, replacing $E_1^d$ by $E_\infty^d$ and the functional $D_1(K^0)$ by $D_\infty(K^0)$ (see (20)—(22)), yield that (9) holds for $p=1$, thus in the problem of uniform approximation for a $W_1$ standard set $K^0$, in both cases $M_0$, $M_1$, there exist simple, sequential $N$-step algorithms, whose global errors are $g_2(K^0)N$ times smaller than those of the optimal passive $N$-step algorithms (again even when we know the values of $x$ only within accuracy of order $N^{-2}$). For $T=[a_1, b_1]\times[a_2, b_2]$ and a corresponding $W_1$ standard set one obtains a two-dimensional $N$-step

algorithm, whose global error, in the $L_\infty$ sense, is less than $g_3(K^0)N^{-2/3}$, by applying in the direction $s_2$ the one-dimensional $L_\infty$ Algorithm 1 and in the direction of $s_1$ the same algorithm for the function $y$ of $s_1$, whose values are the integrals over $T(s_1)$. To prove this we have only to note that for two convex functions $x_1 \geqq x_2$ with derivatives less than $g$

$$\int_a^b (x_1(t)-x_2(t))dt \geqq (2/g)\sup\{x_1(t)-x_2(t) \mid t \in [a, b]\}^2$$

(we can assume that all measurement free subintervals yielded by Algorithm 1 are smaller than $2(b-a)N^{\alpha-1}$), thus the uncertainty in the value of the integral (with respect to $ds_2$) of $y$ for each value of $s_1 \in [a_1, b_1]$ will be of order $N^{-2(1-\alpha)}$, when $\varepsilon_0$ is set to be $N^{-2\alpha}$ $(\alpha = 1/3)$[9].

## REFERENCES


1. Н. С. Бахвалов. Численные методы. Ч. 1. Москва, 1973.
2. C. de Boor, J. Rice. An adaptive algorithm for multivariate approximation. *J. Approx. Theory*, 25, 1979, No 4, 337-360.
3. Gy. Sonnevend. Sequential, optimal order, $N$-step algorithms for the uniform approximation of functions with convex $r$-th derivatives. (Publ. Int. Banach Centre. Warszawa, 1980.)
4. Gy. Sonnevend. Sequential, asymptotically optimal methods for the approximation of convex compact sets in $R^p$ in the Hausdorff metrics. — In: Proc. Conf. on Functions, Series, Operators (In memoriam L. Fejer and F. Riesz) (to appear).
5. R. Schneider. Zur optimalen Approximation konvexer Hyperflächen durch Polyeder.— In: Proc. Conf. Discrete Geometry. Salzburg, 1980.
6. P. M. Gruber, P. Kenderov. Approximation of convex bodies by polytopes. Preprint, 1980.
7. J. F. Traub, H. Wozniakowski. A General Theory of Optimal Algorithms. New York, 1980.
8. Бл. Сендов. Хаусдорфовые приближения. София, 1979.
9. Gy. Sonnevend. Optimal order seqa ential algorithms for the uniform approximation of convex functions over $[0, 1]^2$. *Appl. Math. Optimization*, 10, 1983.



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