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ON THE GENERALIZED ABSOLUTE CESARO-SUMMABILITY AND THE STRONG APPROXIMATION OF ORTHOGONAL SERIES

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Summary. Let $\{\varphi_n(x)\}_{n=0}^{\infty}$ be an orthogonal system on the interval (0, 1). We consider the orthogonal series $\Sigma_{n=0}^{\infty} c_n\varphi_n(x)$ with $\Sigma_{n=0}^{\infty} c_n^2<\infty$. It is well-known that those series converge in L^2 to a square-integrable function $f(x)$. Let us denote the n-th (C, α)-mean $(\alpha>-1)$ of the series by $\sigma_n^{(\alpha)}(x)$.

The series mentioned is said to be $|C, \alpha, \gamma|_{\varkappa}$-summable (at the point x), where $\varkappa\geqq 1$ and $0\leqq\gamma<1$, if the series $\Sigma_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1}|\sigma_n^{(\alpha)}(x)-\sigma_{n-1}^{(\alpha)}(x)|^{\varkappa}$ is convergent. We give sufficient coefficient-conditions for the almost everywhere $|C, \alpha, \gamma|_{\varkappa}$-summability of the series for any system $\{\varphi_n(x)\}_{n=0}^{\infty}$ and to assure the estimation $(\Sigma_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{\nu}^{(\beta)}(x)-f(x)|^k/\Sigma_{\nu=0}^{n}\alpha_{n\nu})^{1/k}=O(n^{-\gamma})$ $(0<k<\gamma^{-1})$ almost everywhere, where $\{\alpha_{n\nu}\}_{n,\nu=0}^{\infty}$ is a triangular matrix general enough.

1. Let $\{\varphi_n\}$ be an orthogonal system on the interval (0, 1). We consider the orthogonal series

$$\sum_{n=0}^{\infty} a_n\varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} a_n^2<\infty. \tag{1}$$

As usual denote by $\sigma_n^{(\alpha)}(x)$ the n-th Cesaro means of order α of (1). The following definition is due to Flett [1]: A series (1) is said to be $|C, \alpha, \gamma|_{\varkappa}$ $(\varkappa\geqq 1,\ \alpha>-1)$ summable at the point x, if the series

$$\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1}|\sigma_n^{(\alpha)}(x)-\sigma_{n-1}^{(\alpha)}(x)|^{\varkappa}$$

is convergent.

In [5] we proved

Theorem A. *Let* $1\leqq\varkappa\leqq 2$ *and* $0\leqq\gamma<1$. *If*

$$\sum_{m=0}^{\infty}\lambda_m\{\sum_{n=2^m+1}^{2^{m+1}}(n^{\gamma}a_n)^2\}^{\varkappa/2}<\infty, \tag{2}$$

where

$$\lambda_m = \begin{cases} 1 & \text{if} \quad \alpha > 1/2, \\ m^{\varkappa/2} & \text{if} \quad \alpha = 1/2, \\ 2^{m\varkappa((1/2)-\alpha)} & \text{if} \quad -1 < \alpha < 1/2, \end{cases}$$

then for any orthonormal system $\{\varphi_n\}$ *on* $(0, 1)$ *the series* (1) *is summable* $|C, \alpha, \gamma|_\varkappa$ *almost everywhere. In the case* $\alpha > 1/2$ *the condition* (2) *is necessary to that the series* (1) *should be summable* $|C, \alpha, \gamma|_\varkappa$ *for any orthonormal system* $\{\varphi_n\}$.

This theorem for $\gamma = 0$ and $\varkappa = 1$ reduces to theorems of Leindler [2]. Very recently Leindler and Schwinn in a joint paper [4] sharpened this special case, proving

Theorem B. *The condition*

$$\sum_{m=0}^{\infty} \{ \sum_{n=\mu_m+1}^{\mu_{m+1}} a_n^2 \}^{1/2} < \infty^{*}, \tag{3}$$

where

$$\mu_m = \mu_m^{(\alpha)} = \begin{cases} 2^{\sqrt{m}} & \text{if} \quad \alpha = 1/2, \\ m^{1/(1-2\alpha)} & \text{if} \quad 0 \leqq \alpha < 1/2, \end{cases}$$

implies the $|C, \alpha, 0|_1$ *summability of series* (1) *for any orthonormal system* $\{\varphi_n\}$ *almost everywhere.*

Using the methods of Leindler and Schwinn we generalize Theorem B and sharpen Theorem A as follows

Theorem 1. *Let* $1 \leqq \varkappa < 2$ *and* $0 \leqq \gamma < 1$. *The condition*

$$\sum_{m=0}^{\infty} \{ \sum_{n=\mu_m+1}^{\mu_{m+1}} (n^\gamma a_n)^2 \}^{\varkappa/2} < \infty \tag{4}$$

with

$$\mu_m = \mu_m^{(\alpha,\varkappa)} = \begin{cases} 2^{m(2-\varkappa)/2} & \text{if} \quad \alpha = 1/2, \\ m^{(2-\varkappa)/(\varkappa(1-2\alpha))} & \text{if} \quad 1 - 1/\varkappa < \alpha < 1/2, \end{cases} \tag{5}$$

implies the $|C, \alpha, \gamma|_\varkappa$ *summability of series* (1) *for any orthonormal system* $\{\varphi_n\}$ *on* $(0, 1)$ *almost everywhere.*

A formal difference between conditions (2) and (4) is that in (2) there is a relatively long block, which is independent of α and $\varkappa$ and the block is multiplied by the factor $\{\lambda_m\}$, depending on α and $\varkappa$. On the other hand, in (4) there are shorter blocks, depending on α and $\varkappa$, but without any factor.

One result of Leindler, proved recently in [3, Theorem 4.1]), shows that the condition (2) implies the condition (4), moreover in the monotonic case they are equivalent, respectively.

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* If in a sum $\sum_{n=p_m}^{p_{m+1}}$ the indices p_m and p_{m+1} are not integers, we extend the summation for all integers $n \in [p_m, p_{m+1}]$.

2. Let us consider a regular summation method, determined by a triangular matrix $(\alpha_{n\nu}/\sum_{\mu=0}^{n}\alpha_{n\mu})_{n,\nu=0}^{\infty}$, where $\alpha_{n\nu}\geqq 0$.

Let now $1<\varkappa\leqq 2$; $0<\gamma<1$; $0<k<\gamma^{-1}$ and suppose that there exist a number $p>1$ and a constant K such that $p'k\geqq\varkappa (p'=p/(p-1))$ and with this p for any $0<\delta<1$ and $2^m<n\leqq 2^{m+1}$

$$\sum_{l=0}^{m}\{\sum_{\nu=2^l-1}^{\min(2^{l+1},n)}\alpha_{n\nu}^{p}(\nu+1)^{p(1-\delta)-1}\}^{1/p}\leqq K(\sum_{\mu=0}^{n}\alpha_{n\mu})n^{-\delta}$$

holds. Finally we define the following notation

$$L=L(\varkappa, p, k)=1/2+1/\varkappa-1/p'k\geqq 1/2. \tag{6}$$

To show a certain connection between the $|C,\alpha,\gamma|_\varkappa$ summability and the strong approximation of the series (1) we proved ([6, Theorem 1])

Theorem C. *Let* $1<\varkappa\leqq 2$; $0<\gamma<1$; $0<k<\gamma^{-1}$ *and*

$$\sum_{m=0}^{\infty}\lambda_m\{\sum_{n=2^m+1}^{2^{m+1}}(n^\gamma a_n)^2\}^{\varkappa/2}<\infty \tag{7}$$

with

$$\lambda_m=\begin{cases}1 & \text{if } \beta>L,\\ m^{\varkappa/2} & \text{if } \beta=L, \qquad\qquad \text{(see (6))}\\ 2^{m\varkappa(L-\beta)} & \text{if } L-1/2<\beta<L,\end{cases}$$

then almost everywhere in $(0, 1)$ *we have*

$$\{(\sum_{\nu=0}^{n}\alpha_{n\nu})^{-1}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{(\beta-1)}(x)-f(x)|^k\}^{1/k}=o_x(n^{-\gamma}), \tag{8}$$

where f is the sum of series (1) *tn* L^2*-sense.*

Now we have

Theorem 2. *If* $1<\varkappa<2$; $0<\gamma<1$; $0<k<\gamma^{-1}$ *and*

$$\sum_{m=0}^{\infty}\{\sum_{n=\mu_m+1}^{\mu_{m+1}}(n^\gamma a_n)^2\}^{\varkappa/2}<\infty \tag{9}$$

with

$$\mu_m=\mu_m^{(\beta,\varkappa)}=\begin{cases}2^{m^{\frac{2-\varkappa}{2}}} & \text{if } \beta=L,\\ m^{\frac{2-\varkappa}{2\varkappa(L-\beta)}} & \text{if } L+1/2-1/\varkappa<\beta<L\end{cases} \qquad \text{(see (6))}$$

then we have the estimation (8) *almost everywhere on* $(0, 1)$, *respectively.*

We mentioned above that (7) implies (9).

3. Proof of Theorem 1. Let $A_m^{(\alpha)}=\binom{m+\alpha}{m}$. Then we have

$$0<c_1(\alpha)\leqq m^{-\alpha}A_m^{(\alpha)}\leqq C_2(\alpha) \qquad (m>0,\ \alpha>-1), \tag{10}$$

$$A_m^{(\alpha)}>0 \qquad (m\geq 0,\ \alpha>-1),$$

$$A_{m+1}^{(\alpha)}>A_m^{(\alpha)} \qquad (m\geq 0,\ \alpha>0),$$

where $c_1(\alpha)$ and $c_2(\alpha)$ are independent of m. Defining

$$L_{n,\nu}^{(\alpha)}=\frac{A_{n+1-\nu}^{(\alpha)}}{A_{n+1}^{(\alpha)}}-\frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}}=\frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}}\,\frac{\nu\alpha}{(n+1-\nu)(n+1+\alpha)},$$

it easily follows that for any $n=1,2,\ldots,\ \nu=0,1,\ldots,\ \alpha>-1,\ \alpha\neq 0$:

(11) $$0<d_1(\alpha)n^{-\alpha-1}(n+1-\nu)^{\alpha-1}\nu\leq L_{n,\nu}^{(\alpha)}\leq d_2(\alpha)n^{-\alpha-1}(n+1-\nu)^{\alpha-1}\nu$$

and $\operatorname{sgn} L_{n,\nu}^{(\alpha)}=\operatorname{sgn}\alpha$, where $d_1(\alpha)$ and $d_2(\alpha)$ are independent of n and ν.

As

$$\sigma_n^{(\alpha)}(x)=(A_n^{(\alpha)})^{-1}\sum_{\nu=0}^{n}A_{n-\nu}^{(\alpha)}a_\nu\varphi_\nu(x),$$

$$\sigma_{n+1}^{(\alpha)}(x)-\sigma_n^{(\alpha)}(x)=\sum_{\nu=0}^{n}L_{n,\nu}^{(\alpha)}a_\nu\varphi_\nu(x)+(A_{n+1}^{(\alpha)})^{-1}a_{n+1}\varphi_{n+1}(x).$$

Let $1\leq\varkappa<2,\ 1-1/\varkappa<\alpha\leq 1/2,\quad 0\leq\gamma<1$ and

$$k_m^{(\alpha,\varkappa,\gamma)}=(\mu_{m+1}^{(\alpha,\varkappa)})^{(2\varkappa\gamma+\varkappa)/(2-\varkappa)}-(\mu_m^{(\alpha,\varkappa)(2\varkappa\gamma+\varkappa)/(2-\varkappa)},$$

where $\mu_m^{(\alpha,\varkappa)}$ is defined by (5).

For the sake of simplicity we use $\mu_m=\mu_m^{(\alpha,\varkappa)}$ and $k_m=k_m^{(\alpha,\gamma,\varkappa)}$. We suppose without loss of generality, that $a_0=a_1=0$. Applying (10), (11) and the Hölder inequality, we have

$$I^{(\alpha,\gamma,\varkappa)}=\sum_{n=2}^{\infty}n^{\varkappa\gamma+\varkappa-1}\int_0^1|\sigma_n^{(\alpha)}(x)-\sigma_{n-1}^{(\alpha)}(x)|^\varkappa dx$$

$$\leq O(1)\sum_{m=0}^{\infty}\sum_{n=\mu_m+1}^{\mu_{m+1}}n^{\varkappa\gamma+\varkappa-1}\Big(\int_0^1|\sigma_{n+1}^{(\alpha)}(x)-\sigma_n^{(\alpha)}(x)|^2dx\Big)^{\varkappa/2}$$

$$\leq O(1)\sum_{m=0}^{\infty}k_m^{(2-\varkappa)/2}\Big(\sum_{n=\mu_m+1}^{\mu_{m+1}}\int_0^1|\sigma_{n+1}^{(\alpha)}(x)-\sigma_n^{(\alpha)}(x)|^2dx\Big)^{\varkappa/2}$$

$$=O(1)\sum_{m=0}^{\infty}k_m^{(2-\varkappa)/2}\Big(\sum_{n=\mu_m+1}^{\mu_{m+1}}\Big(\sum_{\nu=0}^{n}(L_{n,\nu}^{(\alpha)})^2a_\nu^2+(A_{n+1}^{(\alpha)})^{-2}a_{n+1}^2\Big)\Big)^{\varkappa/2}$$

$$=O(1)\sum_{m=0}^{\infty}k_m^{(2-\varkappa)/2}\Big(\sum_{n=\mu_m+1}^{\mu_{m+1}}\sum_{\nu=0}^{n}n^{-2\alpha-2}(n+1-\nu)^{2\alpha-2}\nu^2a_\nu^2\Big)^{\varkappa/2}$$

$$+O(1)\sum_{m=0}^{\infty}k_m^{(2-\varkappa)/2}\Big(\sum_{n=\mu_m+2}^{\mu_{m+1}+1}n^{-2\alpha}a_n^2\Big)^{\varkappa/2}\equiv I_1^{(\alpha,\gamma,\varkappa)}+I_2^{(\alpha,\gamma,\varkappa)}.$$

By the Lagrange theorem we get for $m=1, 2, \ldots$

(12) $$k_m=\begin{cases} O(1)\, m^{-\varkappa/2} \quad 2^{(\varkappa(2\gamma+1)/(2-\varkappa))m^{(2-\varkappa)/2}}, & \text{if } \alpha=1/2, \\ O(1)\ \ m^{2(\gamma+\alpha)/(1-2\alpha)}, & \text{if } 1-1/\varkappa<\alpha<1/2. \end{cases}$$

By (12), (5) and (4) we have

$$I_2^{(1/2,\gamma,\varkappa)} = O(1) \sum_{m=1}^{\infty} m^{\varkappa(\varkappa-2)/4}\, 2^{\varkappa(\gamma+1/2)m^{(2-\varkappa)/2}} \Big(\sum_{n=\mu_m+1}^{\mu_{m+1}} n^{-1} a_n^2\Big)^{\varkappa/2}$$

$$\leqq O(1) \sum_{m=1}^{\infty} \Big(\sum_{n=\mu_m+1}^{\mu_{m+1}} n^{2\gamma} a_n^2\Big)^{\varkappa/2} < \infty .$$

If $1-1/\varkappa<\alpha<1/2$ by (12), (5) and (4), we obtain

$$I_2^{(\alpha,\gamma,\varkappa)} = O(1) \sum_{m=1}^{\infty} m^{(\gamma+\alpha)\,(2-\varkappa)/(1-2\alpha)} \Big(\sum_{n=\mu_m+1}^{\mu_{m+1}} n^{-2\alpha} a_n^2\Big)^{\varkappa/2}$$

$$= O(1) \sum_{m=1}^{\infty} \Big(\sum_{n=\mu_m+1}^{\mu_{m+1}} n^{2\gamma} a_n^2\Big)^{\varkappa/2} < \infty .$$

Using $\mu_0=0$ for $1-1/\varkappa<\alpha\leqq 1/2$, we have the estimation

$$I_1^{(\alpha,\gamma,\varkappa)} \leqq O(1) \sum_{m=2}^{\infty} \Big\{ k_m^{(2-\varkappa)/\varkappa} \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{k=0}^{m-2} \sum_{\nu=\mu_k+1}^{\mu_{k+1}} n^{-2\alpha-2} (n+1-\nu)^{2\alpha-2} \nu^2 a_\nu^2 \Big\}^{\varkappa/2}$$

$$+ O(1) \sum_{m=1}^{\infty} \Big\{ k_m^{(2-\varkappa)/\varkappa} \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{\nu=\mu_{m-1}+1}^{\mu_m} n^{-2\alpha-2} (n+1-\nu)^{2\alpha-2} \nu^2 a_\nu^2 \Big\}^{\varkappa/2}$$

$$+ O(1) \sum_{m=0}^{\infty} \Big\{ k_m^{(2-\varkappa)/\varkappa} \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{\nu=\mu_m+1}^{n} n^{-2\alpha-2} (n+1-\nu)^{2\alpha-2} \nu^2 a_\nu^2 \Big\}^{\varkappa/2}$$

$$= I_{11}^{(\alpha,\gamma,\varkappa)} + I_{12}^{(\alpha,\gamma,\varkappa)} + I_{13}^{(\alpha,\gamma,\varkappa)}.$$

Easy to see that

$$\sum_{n=\mu_m+1}^{\mu_{m+1}} (n+1-\mu_m)^{2\alpha-2} = \begin{cases} O(\log \mu_m) & \text{if } \alpha=1/2, \\ O(1) & \text{if } 1-1/\varkappa<\alpha<1/2. \end{cases}$$

Using this fact

$$I_{12}^{(\alpha,\gamma,\varkappa)} = O(1) \sum_{m=1}^{\infty} \mu_m^{-\varkappa\alpha} k_m^{(2-\varkappa)/2} \Big\{ \sum_{n=\mu_m+1}^{\mu_{m+1}} (n+1-\mu_m)^{2\alpha-2} \Big\}^{\varkappa/2} \Big\{ \sum_{\nu=\mu_{m-1}+1}^{\mu_m} a_\nu^2 \Big\}^{\varkappa/2}$$

by (12), (5), (4) we get for $\alpha=1/2$

$$I_{12}^{(1/2,\,\gamma,\varkappa)} = O(1) \sum_{m=1}^{\infty} 2^{\varkappa\gamma m^{(2-\varkappa)/2}} \Big\{ \sum_{\nu=\mu_m+1}^{\mu_{m+1}} a_\nu^2 \Big\}^{\varkappa/2} < \infty$$

and by (12), (5), (4) for $1-1/\varkappa \leq \alpha < 1/2$

$$I_{12}^{(\alpha,\gamma,\varkappa)} = O(1) \sum_{m=1}^{\infty} m^{(\gamma(2-\varkappa))/(1-2\alpha)} \{ \sum_{n=\mu_m+1}^{\mu_{m+1}} a_\nu^2 \}^{\varkappa/2} < \infty.$$

As $\mu_m < \nu \leq \mu_{m+1}$, we have

$$\sum_{n=\nu}^{\mu_{m+1}} (n+1-\nu)^{2\alpha-2} \leq \begin{cases} O(1) \log \mu_m & \text{if } \alpha = 1/2, \\ O(1) & \text{if } 1-1/\varkappa < \alpha < 1/2. \end{cases}$$

Using (12), (10), (5) and (4), we get for $1-1/\varkappa < \alpha \leq 1/2$

$$I_{13}^{(\alpha,\gamma,\varkappa)} = O(1) \sum_{m=0}^{\infty} k_m^{(2-\varkappa)/2} \{ \sum_{\nu=\mu_m+1}^{\mu_{m+1}} \sum_{n=\nu}^{\mu_{m+1}} n^{-2\alpha-2} (n+1-\nu)^{2\alpha-2} \nu^2 a_\nu^2 \}^{\varkappa/2}$$

$$\leq O(1) \sum_{m=0}^{\infty} \mu_m^{-\varkappa\alpha} k_m^{(2-\varkappa)/2} \{ \sum_{n=\nu}^{\mu_{m+1}} (n+1-\nu)^{2\alpha-2} \}^{\varkappa/2} \{ \sum_{\nu=\mu_m+1}^{\mu_{m+1}} a_\nu^2 \}^{\varkappa/2} < \infty.$$

It remains to prove that

$$I_{11}^{(\alpha,\gamma,\varkappa)} < \infty \quad (1-1/\varkappa < \alpha \leq 1/2).$$

First we consider the case $\alpha = 1/2$:

$$I_{11}^{(1/2,\gamma,\varkappa)} \leq O(1) \sum_{m=2}^{\infty} \mu_m^{-3\varkappa/2} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} \sum_{k=0}^{m-2} (\mu_m - \mu_{k+1})^{-\varkappa/2} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+2}^{\infty} \mu_m^{-3\varkappa/2} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} (\mu_m - \mu_{k+1})^{-\varkappa/2}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+2}^{k+k^{\varkappa/2}} \mu_m^{-3\varkappa/2} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} (\mu_m - \mu_{k+1})^{-\varkappa/2}$$

$$+ O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+k^{\varkappa/2}+1}^{\infty} \mu_m^{-3\varkappa/2} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} (\mu_m - \mu_{k+1})^{-\varkappa/2}$$

$$= I_{111}^{(1/2,\gamma,\varkappa)} + I_{112}^{(1/2,\gamma,\varkappa)}.$$

Using the Lagrange theorem by (5) after a usual computation, we can see that

$$(13) \quad \mu_m - \mu_{k+1} \geq \begin{cases} (1-\varkappa/2)(\ln 2) 2^{-\varkappa/2} k^{-\varkappa/2} (m-k-1) \mu_{k+1} & \text{if } k+2 \leq m \leq k+k^{\varkappa/2}, \\ (1-2^{-(1-\varkappa/2)\,3^{-\varkappa/2}}) \mu_m & \text{if } k+k^{\varkappa/2}+1 \leq m < \infty, \end{cases}$$

and

$$(14) \quad \mu_m^{-3\varkappa/2} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} = O(1)\, m^{-\varkappa/2} \mu_m^{\varkappa(\gamma-1/2)}.$$

By (13), (14) and (4) we have

$$I_{111}^{(1/2,\gamma,\varkappa)} \leqq O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+2}^{k+k^{\varkappa/2}} m^{-\varkappa/2} \mu_m^{\varkappa(\gamma-1/2)} \mu_k^{-\varkappa/2} k^{\varkappa^2/4} (m-k)^{-\varkappa/2}$$

$$\leqq O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa\gamma} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} k^{\varkappa^2/4-\varkappa/2} \sum_{m=k+2}^{k+k^{\varkappa/2}} (m-k)^{-\varkappa/2}$$

$$= O(1) \sum_{k=0}^{\infty} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} < \infty,$$

by (13), (14) and (4)

$$I_{112}^{(1/2,\gamma,\varkappa)} \leqq O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+k^{\varkappa/2}+1}^{\infty} m^{-\varkappa/2} \mu_m^{\varkappa(\gamma-1)}$$

$$\leqq O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+1}^{\infty} m^{-\varkappa/2} 2^{\varkappa(\gamma-1)} m^{(2-\varkappa)/2}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} 2^{\varkappa(\gamma-1)} k^{(2-\varkappa)/2} < \infty.$$

Now we consider the case $1-1/\varkappa < \alpha < 1/2$.

$$I_{11}^{(\alpha,\gamma,\varkappa)} \leqq O(1) \sum_{m=2}^{\infty} \mu_m^{-\varkappa\alpha-\varkappa} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} \sum_{k=0}^{m-2} (\mu_m-\mu_{k+1})^{\varkappa\alpha-\varkappa} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+2}^{\infty} \mu_m^{-\varkappa\alpha-\varkappa} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} (\mu_m-\mu_{k+1})^{\varkappa\alpha-\varkappa}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=k+2}^{2k+1} \mu_m^{-\varkappa\alpha-\varkappa} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} (\mu_m-\mu_{k+1})^{\varkappa\alpha-\varkappa}$$

$$+ O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=2k+2}^{\infty} \mu_m^{-\varkappa\alpha-\varkappa} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} (\mu_m-\mu_{k+1})^{\varkappa\alpha-\varkappa}$$

$$= I_{111}^{(\alpha,\gamma,\varkappa)} + I_{112}^{(\alpha,\gamma,\varkappa)}.$$

Easy to see that

$$(15) \qquad \mu_m - \mu_{k+1} \geqq \begin{cases} \dfrac{2-\varkappa}{\varkappa(1-2\alpha)} \dfrac{m-k-1}{k+1} \mu_{k+1} & \text{if } k+2 \leqq m \leqq 2k+1, \\ (1-2^{(\varkappa-2)/(\varkappa(1-2\alpha))}) \mu_m & \text{if } 2k+2 \leqq m < \infty, \end{cases}$$

and

$$(16) \qquad \mu_m^{-\varkappa\alpha-\varkappa} k_m^{(2-\varkappa)/2} (\mu_{m+1}-\mu_m)^{\varkappa/2} = O(1)\ m^{(\varkappa\alpha-1)/(1-2\alpha)} \mu_m^{\varkappa\gamma}.$$

If $k+2 \leqq m \leqq 2k+1$, then $\mu_m = O(\mu_k)$. By (15), (16) and (4) we have

$$I_{111}^{(\alpha,\gamma,\varkappa)} \leq O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa(\gamma+\alpha)} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} k^{(\varkappa\alpha-1)/(1-2\alpha)+(\varkappa(1-\alpha))} \sum_{m=k+2}^{2k+1} (m-k)^{-\varkappa+\varkappa\alpha}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa(\gamma+\alpha)} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} k^{(\varkappa\alpha-2\alpha)/(1-2\alpha)} < \infty .$$

In the case $2k+2 \leq m < \infty$, using (15), (16) and (4), we obtain

$$I_{112}^{(\alpha,\gamma,\varkappa)} \leq O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=2k+2}^{\infty} m^{(\varkappa\alpha-1)/(1-2\alpha)} \mu_m^{\varkappa\gamma-\varkappa+\varkappa\alpha}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} \sum_{m=2k+2}^{\infty} m^{(\varkappa\gamma-\varkappa)(2-\varkappa)/(\varkappa(1-2\alpha))-1}$$

$$= O(1) \sum_{k=0}^{\infty} \mu_k^{\varkappa\gamma} \{ \sum_{\nu=\mu_k+1}^{\mu_{k+1}} a_\nu^2 \}^{\varkappa/2} < \infty .$$

Collecting our estimations, we get

$$(17) \qquad \sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} \int_0^1 |\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x)|^\varkappa \, dx \leq O(1) \sum_{m=0}^{\infty} \{ \sum_{n=\mu_m+1}^{\mu_{m+1}} (n^\gamma a_n)^2 \}^{\varkappa/2}$$

for $1 \leq \varkappa < 2$, $1 - 1/\varkappa < \alpha \leq 1/2$ and $0 \leq \gamma < 1$. Applying the Levi theorem, we obtain that the series

$$\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} |\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x)|^\varkappa$$

converges almost everywhere.

4. The proof of Theorem 2 wholly follows the line of the proof of Theorem C, but we have to exchange the Lemma 2 of [6] for the following

Lemma. *If* $0 \leq \gamma < 1$; $1 \leq \varkappa < 2$ *and* $1 - 1/\varkappa < \alpha \leq 1/2$, *then*

$$\int_0^1 \Big(\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} |\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x)|^\varkappa \Big) dx \leq O(1) \sum_{m=0}^{\infty} \{ \sum_{n=\mu_m+1}^{\mu_{m+1}} (n^\gamma a_n)^2 \}^{\varkappa/2},$$

where μ_m *is determined by* (5).

This lemma is given by (17) immediately.

REFERENCES

1. T. M. Flett. Some more theorems concerning the absolute summability of Fourier series and power series. *Proc. London Math. Soc.*, **8**, 1958, 357-387.
2. L. Leindler. Über die absolute Summierbarkeit der Orthogonalreihen. *Acta Sci. Math.*, **22**, 1961, 243-268.
3. L. Leindler. On relations of coefficient conditions. *Acta Math. Acad. Sci. Hung.*, **39**, 1982, 409-420.
4. L. Leindler, H. Schwinn. Über die absolute Summierbarkeit der Orthogonalreihen. *Acta Sci. Math.*, **43**, 1981, 311-319.
5. I. Szalay. On generalized absolute Cesaro summability of orthogonal series. *Acta Sci. Math.*, **32**, 1971, 51-57.
6. I. Szalay. On the strong approximation of orthogonal series. — In: Constructive Function Theory'77. Sofia, 1980, 505-510.

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