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NEW ESTIMATES FOR THE HAUSDORFF AND LOCAL APPROXIMATIONS OF FUNCTIONS

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Summary. In this paper new estimates for the Hausdorff and local approximations of functions by polynomials are obtained. These estimates are expressed in terms of the modulus of continuity of the estimated function f and modulus of continuity of the function θ_f, where $\theta_f(x)=\operatorname{arctg} D(f;\ x)$ and $D(f;\ x)$ is the segment derivative of the function f in the point x (see [2]).

1. Introduction. We shall use the following notations:

T_n and P_n are the set of all trigonometric and algebraic polynomials of degree $\leq n$.

$HE(2\pi,\ T_n;\ f)$ is the best Hausdorff approximation of function f by trigonometric polynomials of degree $\leq n$, i. e.

$$HE(2\pi,\ T_n;\ f)=\inf\{r(f,\ \tau):\ \tau\in T_n\},$$

where $r(f,\ \tau)$ is the Hausdorff distance between functions f and τ with parameter 1 (see [1, 4]).

$HE(\Delta,\ P_n;\ f)$ is the best Hausdorff approximation of function f by algebraic polynomials of degree $\leq n$ on the interval Δ.

Let f be a function defined on the interval Δ. We shall consider the function θ_f, which is $\theta_f(x)=\operatorname{arctg} D(f;\ x)$, $x\in\Delta$, and $D(f;\ x)$ is the segment derivative of the function f in the point x (see [2]). It should be noted that, if there exists a usual derivative $f'(x)$, then $D(f;\ x)=f'(x)$ and by definition $\theta_f(x)=\operatorname{arctg} f'(x)$. In the other case $D(f;\ x)$ is the smallest interval $[a,\ b]$ containing all derivative numbers of the function f in the point x. In particular, this interval may be infinite. In this case $\theta_f(x)$ is the interval $[\operatorname{arctg} a,\ \operatorname{arctg} b]$. But the interesting case is when θ_f is a continuous function.

The local modulus of continuity of a set valued function f, defined on the interval Δ in the point x, is

$$\omega(f, x;\ \delta)=\sup\{|t_1-t_2|:\ t_i\in f(x_i),\ x_i\in[x-\delta,\ x+\delta]\cap\Delta,\quad i=1,\ 2\}.$$

Thus, the modulus of continuity of the function f is

$$\omega(f;\ \delta)=\sup\{\omega(f,\ x;\ \delta):\ x\in\Delta\}.$$

Sendov and Popov [3] obtained the following result: For every function $f \in C_{2\pi}$ the estimate

(1) $$HE(2\pi,\ T_n;\ f) = O(\ln(e+n\omega(f;\ 1/n))/n + O(1/n)$$

is true. Analogous result is true in the algebraic case: For every continuous function f, defined on the interval Δ, the following estimate is true:

(2) $$HE(\Delta,\ P_n;\ f) = O(\ln(e+n\omega(f;\ 1/n))/n) + O(1/n).$$

On the other hand, Sendov [4] received

(3) $$HE([-1,\ 1],\ P_n;\ \sigma_\alpha) = O(1/n),$$

where $\sigma_\alpha(x) = |x|^\alpha \operatorname{sign} x$, $x \in [-1,\ 1]$ for every α, $0<\alpha<1$.

We note that from (2) does not follow (3), because $\omega(\sigma_\alpha,\ 1/n) = n^{-\alpha}$ and (2) implies $HE([-1,\ 1],\ P_n;\ \sigma_\alpha) = O(\ln n/n)$.

The purpose of this report is to obtain a new estimate of type (2), from which follows the estimate (3) of the function σ_α particularly. A similar result is obtained with respect to the local approximation of functions.

2. Best Hausdorff Approximation

Theorem 1. *For every 2π-periodical, bounded function f the following estimate holds true:*

(4) $$HE(2\pi,\ T_n;\ f) \le C\ln(e+n\omega(f;\ 1/n)\omega(\theta_f;\ 1/n))/n$$

where $C>0$ is an absolute constant.

Theorem 2. *For every bounded function f, defined on the interval Δ, the estimate*

(5) $$HE(\Delta,\ P_n;\ f) \le C\ln(e+n\omega(f;\ 1/n)\omega(\theta_f;\ 1/n))/n$$

is true, where $C>0$ is an absolute constant.

We shall prove only Theorem 1.

Lemma 1. *For every function f, defined on the interval $[x-t,\ x+t]$, where $t \ge \delta > 0$, the inequalities*

(6) $$|f(x+t)-2f(x)+f(x-t)| \le 2t\omega(\theta_f;\ t)(1+\omega(f;\ t)/t)(1+|f(x+\delta)-f(x)|/\delta),$$

(7) $$|f(x+t)-2f(x)+f(x-t)| \le 2t\omega(\theta_f;\ t)(1+\omega(f;\ t)/t)(1+|f(x-\delta)-f(x)|/\delta)$$

are true.

Proof. We shall prove the inequality (6). The inequality (7) may be proved analogously. Let we denote $\alpha = \operatorname{arctg}[(f(x+t)-f(x))/t]$, $\beta = \operatorname{arctg}(f(x+\delta)-f(x))/\delta$ and $\gamma = \operatorname{arctg}(f(x)-f(x-t))/t$. Then we have

(8) $$|f(x+t)-2f(x)+f(x-t)| = t|(f(x+t)-f(x))/t-(f(x)-f(x-t))/t|$$

$$= t|\operatorname{tg}\alpha - \operatorname{tg}\gamma| \le t|\operatorname{tg}\alpha - \operatorname{tg}\beta| + t|\operatorname{tg}\beta - \operatorname{tg}\gamma|$$

$$= t|\sin(\alpha-\beta)|(1+\operatorname{tg}^2\alpha)^{1/2}(1+\operatorname{tg}^2\beta)^{1/2} + t|\sin(\beta-\gamma)|(1+\operatorname{tg}^2\beta)^{1/2}(1+\operatorname{tg}^2\gamma)^{1/2}$$

$$\le t|\alpha-\beta|(1+|\operatorname{tg}\alpha|)(1+|\operatorname{tg}\beta|) + t|\beta-\gamma|(1+|\operatorname{tg}\beta|)(1+|\operatorname{tg}\gamma|).$$

By the Lagrange formula (see [2]) the inclusions

$$(f(x+t)-f(x))/t \subset D(f;\ [x,\ x+t]),$$

$$(f(x)-f(x-t))/t \subset D(f;\ [x-t,\ x]),$$
$$(f(x+\delta)-f(x))/\delta \subset D(f;\ [x,\ x+\delta])$$

are true. From here we obtain $|\alpha-\beta|\leq\omega(\theta_f;\ t)$ and $|\beta-\gamma|\leq\omega(\theta_f;\ t)$. Besides, the inequalities $|f(x+t)-f(x)|\leq\omega(f;\ t)$ and $|f(x)-f(x-t)|\leq\omega(f;\ t)$ hold true. Thus, (8) implies (6). The lemma is proved.

Lemma 2. *For every bounded, 2π-periodic function f and for every integer number q, $q\geq 1$, there exists a function g, $g\in C_{2\pi}$, such that*

1) $r(f,\ g)\leq 2\pi/q$,
2) $\omega(g;\ \delta)\leq\omega(f;\ 2\pi/q)$ *for* $\delta\leq\pi/(4q)$,
3) $\omega(\theta_g;\ \delta)\leq 22\omega(\theta_f;\ 4\pi/q)$ *for* $\delta\leq\pi/(4q)$,
4) *g is $\pi/(2q)$-monotone function, what is g is monotone function on every interval with a length $\leq\pi/(2q)$.*

Proof. Let us denote

$$x_k=2\pi k/q,\quad k=0,\ \pm1,\ \pm2,\ldots,$$
$$m_k=\inf\{f(x): x\in[x_k,\ x_{k+1}]\},$$
$$M_k=\sup\{f(x): x\in[x_k,\ x_{k+1}]\}.$$

Let us consider the continuous function $g(x)$, defined as follows:

a) If $0\,\overline{\in}\,\theta_f(x)$, $x\in[x_{k-1},\ x_{k+2}]$, we set

$$g(x)=f(x_k)+(x-x_k)(f(x_{k+1})-f(x_k))/(x_{k+1}-x_k).$$

b) If $0\in\theta_f(x)$, $x_k\in[x_{k-1},\ x_{k+2}]$ and $f(x_k)-f(x_{k-1})\geq 0$, we set

$$g(x)=\begin{cases} m_k & \text{for } x\in[x_k, x_k+(x_{k+1}-x_k)/4],\\ M_k & \text{for } x\in[x_k+(x_{k+1}-x_k)/2,\ x_k+3(x_{k+1}-x_k)/4],\\ \text{linear} & \text{for } x\in[x_k+(x_{k+1}-x_k)/4,\ x_k+(x_{k+1}-x_k)/2]\\ & \quad\cup[x_k+3(x_{k+1}-x_k)/4,\ x_{k+1}].\end{cases}$$

c) If $0\in\theta_f(x)$, $x\in[x_{k-1},\ x_{k+2}]$ and $f(x_k)-f(x_{k-1})<0$, we set

$$g(x)=\begin{cases} M_k & \text{for } x\in[x_k,\ (x_{k+1}-x_k)/4],\\ m_k & \text{for } x\in[x_k+(x_{k+1}-x_k)/2,\ x_k+3(x_{k+1}-x_k)/4],\\ \text{linear} & \text{for } x\in[x_k+(x_{k+1}-x_k)/4,\ x_k+(x_{k+1}-x_k)/2]\\ & \quad\cup[x_k+3(x_{k+1}-x_k)/4,\ x_{k+1}].\end{cases}$$

By the definition of g one obtains immediately 1) and 4).

For the proof of 2) we denote $d=\max_k\max[|M_k-m_k|,\ |M_k-m_{k+1}|,\ |M_k-m_{k-1}|]$. Obviously we have $\omega(f;\ 2\pi/q)\geq\delta$. From the definition of g it follows $\omega(g;\ \delta)\leq 4dq\delta/\pi$ for $\delta\leq\pi/(8q)$. Consequently

$$\omega(g;\ \delta)\leq 4q\delta/\pi\omega(f;\ 2\pi/q)\leq\omega(f;\ 2\pi/q).$$

For the proof of 3) we suppose that for $\delta<\pi/(4q)$ we have $\omega(\theta_g;\ \delta)=|\theta_g(\xi)-\theta(\eta)|>0$, where $0<\eta-\xi\leq\pi/(4q)$, $\xi,\ \eta\in[x_k,\ x_{k+2}]$ for some k. The proof is evident, if $\omega(\theta_g;\ \delta)=0$ for $\delta<\pi/(4q)$. We shall consider 3 cases:

i) $0\,\overline{\in}\,\theta_f(x)$ for $x\in[x_{k-1},\ x_{k+3}]$.

By the assumptions it follows that $\xi\in[x_k,\ x_{k+1})$, $\eta\in(x_{k+1},\ x_{k+2}]$. On the other hand, by the Lagrange theorem it follows

$$(f(x_{k+1})-f(x_k))/(x_{k+1}-x_k)\subset D(f;\ [x_k,\ x_{k+1}]),$$
$$(f(x_{k+2})-f(x_{k+1}))/(x_{k+2}-x_{k+1})\subset D(f;\ [x_{k+1},\ x_{k+2}]).$$

Since $\theta_g(\xi)=(f(x_{k+1})-f(x_k))/(x_{k+1}-x_k)$ and $\theta_g(\eta)=(f(x_{k+2})-f(x_{k+1}))/(x_{k+2}-x_{k+1})$, we have $\omega(\theta_g;\ \delta)\leq\omega(\theta_f;\ 4\pi/q)$.

ii) $0\in\theta_f(x)$ for $x\in[x_{k-1},\ x_{k+2}]$, but $0\notin\theta_f(x)$ for $x\in[x_{k+2},\ x_{k+3}]$ and $\xi\in[x_k,\ x_{k+1})$ (The case $\xi\in[x_{k+1},\ x_{k+2}]$ is the same like iii). By the definition of g it follows $\theta_g(\eta)=0$ $(\eta\leq\xi+\pi/(4q))$. By Lagrange theorem it follows the existence of $t\in[x_k,\ x_{k+1}]$ such that $\theta_g(\xi)\in\theta_f(t)$. Since $0\notin\theta_f(x)$ for $x\in[x_{k+2},\ x_{k+3}]$, we have $\omega(\theta_g;\ \delta)\leq\omega(\theta_f;\ 4\pi/q)$.

iii) $0\notin\theta_f(x)$, $x\in[x_{k-1},\ x_{k+2}]$.

By the definition of g it follows that $\theta_g(\xi)=0$ or $\theta_g(\eta)=0$. We suppose that $\theta_g(\xi)\neq0$ and $\theta_g(\eta)=0$. Thus $\theta_g(\xi)=\operatorname{arctg}4qc/\pi$, where $c=\max\{|M_k-m_k|,\ |M_k-m_{k+1}|,\ |M_k-M_{k+1}|,\ |m_k-m_{k+1}|\}$. Evidently, there exist two points $t_1,\ t_2\in[x_k,\ x_{k+2}]$ such that $c=|f(t_1)-f(t_2)|$. By Lagrange theorem there exists a point $z\in[t_1,\ t_2]$ such that $\operatorname{arctg}(f(t_1)-f(t_2))/(t_1-t_2)\in\theta_f(z)$. Since $|t_1-t_2|\leq4\pi/q$, the inequality $|\operatorname{arctg}(f(t_1)-f(t_2))/(t_1-t_2)|\geq\operatorname{arctg}qc/4\pi$ is true, i. e. $\omega(\theta_f;\ 4\pi/q)\geq\operatorname{arctg}qc/4\pi$. This implies 3), because $22\operatorname{arctg}t\geq\operatorname{arctg}16t$ for all $t\geq0$.

Lemma 3. *Let m, r be positive integers ($r\geq3$), $\delta>0$ and*

$$\lambda_{m,r}\int_{-\pi}^{\pi}(\sin(mt/2)/m\sin(t/2))^{2r}dt=1.$$

Then the inequality

$$\lambda_{m,r}\int_{\delta}^{\pi}t^2\left(\frac{\sin mt/2}{m\sin t/2}\right)^{2r}dt\leq\frac{\pi\delta^2}{4(2r-3)}\left(\frac{\pi}{2m\delta}\right)^{2r-1}\tag{9}$$

holds true.

Proof. At first we shall estimate $\lambda_{m,r}$. We have

$$\lambda_{m,r}^{-1}=\int_{-\pi}^{\pi}\left(\frac{\sin mt/2}{m\sin t/2}\right)^{2r}dt\geq2\int_0^{\pi/m}\left(\frac{\sin mt/2}{m\sin t/2}\right)^{2r}dt$$

$$\geq2\int_0^{\pi/m}\left(\frac{mt/\pi}{mt/2}\right)^{2r}dt=2^{2r+1}\pi^{-2r+1}m^{-1}.$$

Then we have

$$\lambda_{m,r}\int_{\delta}^{\pi}t^2\left(\frac{\sin mt/2}{m\sin t/2}\right)^{2r}dt\leq\lambda_{m,r}\int_{\delta}^{\pi}t^2(m\sin t/2)^{-2r}dt$$

$$\leq\lambda_{m,r}(\pi/m)^{2r}\int_{\delta}^{\infty}t^{-2r+2}dt\leq\frac{\pi}{4}\ \frac{\delta^2}{2r-3}\left(\frac{\pi^2}{2m\delta}\right)^{2r-1}.$$

Lemma 4. *Let g be continuous 2π-periodic and 2δ-monotone function. If the inequality*

$$\omega(\theta_g;\ \delta)\frac{\pi}{2r-3}\left(\frac{\pi^2}{2m\delta}\right)^{2r-1}.(2\omega(g;\ \delta)/\delta+1)\leq\frac{1}{2}\tag{10}$$

holds, where m, r are positive integers, $r\geq 3$ *and* $\delta>0$, *then*

$$r(g,\ L_{m,r}(g))\leq 3\delta/2, \tag{11}$$

where

$$L_{m,r}(g;\ x)=\int_{-\pi}^{\pi} g(x+t)K_{m,r}(t)dt\ ; \qquad \int_{-\pi}^{\pi} K_{m,r}(t)dt=1$$

and $K_{m,r}(t)=\lambda_{m,r}\left(\frac{\sin mt/2}{m\sin t/2}\right)^{2r}$.

Proof. It is sufficient to prove the following inequalities:

$$L_{m,r}(g;\ x)\leq S(g,\ x,\ \delta)+\delta/2 \text{ for all } x, \tag{12}$$

$$L_{m,r}(g;\ x)\geq I(g,\ x,\ \delta)-\delta/2 \text{ for all } x, \tag{13}$$

where

$$S(g,\ x,\ \delta)=\sup\{g(t):\ t\in[x-\delta,\ x+\delta]\},$$
$$I(g,\ x,\ \delta)=\inf\{g(t):\ t\in[x-\delta,\ x+\delta]\}.$$

We shall prove only (12). The inequality (13) is proved analogously. Since the function g is 2δ-monotone, we may assume that g is an increasing function in the interval $[x-\delta,\ x+\delta]$. Then we have $S(g,\ x,\ \delta)=g(x+\delta)$. Consequently we have

$$L_{m,r}(g;\ x)=g(x)+\int_{-\pi}^{\pi}(g(x+t)-g(x))K_{m,r}(t)dt$$

$$=g(x)+\int_{-\delta}^{0}(g(x+t)-g(x))K_{m,r}(t)dt$$

$$+\int_{0}^{\delta}(g(x+t)-g(x))K_{m,r}(t)\,dt+\int_{\delta}^{\pi}(g(x+t)-2g(x)+g(x-t))K_{m,r}(t)dt$$

$$\leq g(x)+\int_{0}^{\delta}(g(x+\delta)-g(x))K_{m,r}(t)\,dt+\int_{\delta}^{\pi}|g(x+t)-2g(x)+g(x-t)|K_{m,r}(t)dt$$

$$\leq g(x)+(g(x+\delta)-g(x))/2+\int_{\delta}^{\pi}|g(x+t)-2g(x)+g(x-t)|K_{m,r}(t)dt.$$

From the inequalities (6), (9), (10) and the property $\omega(g;\ t)/t\leq 2\omega(g;\ \delta)/\delta,\ t\geq\delta$' of the modulus of continuity we obtain

$$\int_{\delta}^{\pi}|g(x+t)-2g(x)+g(x-t)|K_{m,r}(t)dt$$

$$\leq 2\left(1+\frac{g(x+\delta)-g(x)}{\delta}\right)\int_{\delta}^{\pi}t\omega(\theta_g;\ t)\left(1+\frac{\omega(g;\ t)}{t}\right)K_{m,r}(t)dt$$

$$\leq 2\left(1+\frac{g(x+\delta)-g(x)}{\delta}\right)2\frac{\omega(\theta_g;\ \delta)}{\delta}\left(1+2\frac{\omega(g;\ \delta)}{\delta}\right)\int_{\delta}^{\pi}t^2K_{m,r}(t)dt$$

$$\leq\left(1+\frac{g(x+\delta)-g(x)}{\delta}\right)\omega(\theta_g;\ \delta)\left(1+2\,\frac{\omega(g;\ \delta)}{\delta}\right)\frac{\pi\delta}{2r-3}\left(\frac{\pi^2}{2m\delta}\right)^{2r-1}\leq\frac{\delta}{2}+\frac{g(x+\delta)-g(x)}{2}.$$

Then the last two inequalities imply (12).

Proof of Theorem 1. Let us denote $r=[\ln(e^2+Cn\omega(f;\ 1/n)\omega(\theta_f;\ 1/n))/2]$, $m=[n/r]$, $\delta=\pi^2/(2me)$, $q=[\pi/(8\delta)]$, where $C>0$ is a sufficiently large constant. Then for $n\geq 2r(1+\pi e)$ we have

$$2\pi/q\leq 4\pi^2 er/n, \tag{14}$$

$$\delta^{-1}\leq 2n/(e\pi^2 r). \tag{15}$$

By Lemma 2 and Lemma 4 there exists a function g (g is 2δ-monotone function) such that

$$r(f;\ g)\leq 2\pi/q \quad\text{and}\quad r(g;\ L_{m,r}(g))\leq 3\delta/2, \tag{16}$$

if the inequality (10) holds true. Thus Theorem 1 follows from (14) and (16), because $r(f;\ L_{m,r}(g))\leq r(f;\ g)+r(g;\ L_{m,r}(g))$ and $L_{m,r}(g)$ is a trigonometric polynomial of degree $\leq n$. It remains to show that there exists a constant $C>0$ such that (10) holds true. For this it is enough to show the inequalities

$$\omega(\theta_g;\ \delta)\frac{\pi}{2r-3}\left(\frac{\pi^2}{2m\delta}\right)^{2r-1}\leq\frac{1}{4}, \tag{17}$$

$$\omega(\theta_g;\ \delta)\omega(g;\ \delta)\,\frac{2\pi}{(2r-3)\delta}\left(\frac{\pi^2}{2m\delta}\right)^{2r-1}\leq\frac{1}{4}. \tag{18}$$

The first one follows from the inequalities $\omega(\theta_g;\ \delta)\leq\pi$, $r\geq 3$ and $(\pi/2m\delta)^{2r-1}\leq e^{-r}$. We shall check (18). Lemma 2 implies

$$\omega(\theta_g;\ \delta)\leq 22\,\omega(\theta_f;\ 4\pi/q)\ \text{ and }\ \omega(g;\ \delta)\leq\omega(f;\ 2\pi/q).$$

Then, using (14), (15) and the property $\omega(f;\ \lambda\delta)\leq(\lambda+1)\omega(f;\ \delta)$ of the modulus of continuity, we have

$$\omega(\theta_g;\ \delta)\omega(g;\ \delta)\,\frac{2\pi}{(2r-3)\delta}\left(\frac{\pi^2}{2m\delta}\right)^{2r-1}\leq 22\,\omega\left(\theta_f;\ \frac{4\pi}{q}\right)\omega\left(f;\ \frac{2\pi}{q}\right)$$

$$\times\frac{2\pi}{(2r-3)\delta}\cdot e^{-2r+1}\leq\frac{44\,\pi\,2n2r^2(4\pi^2e+1)^2\omega(\theta_f;\ n^{-1})\omega(f;\ n^{-1})}{e\pi^2 r(2r-3)\,.\,C\,.\,n\omega(f;\ n^{-1})\omega(\theta_f;\ n^{-1})}$$

$$\leq 176(4\pi^2e+1)^2/(e\pi C).$$

This implies (18) with $C\geq 704(4\pi^3e+1)^2/e\pi$, which completes the proof.

Corollary 1. *Let us consider the function* $\sigma_\alpha(x)=|x|^\alpha\operatorname{sign}x$, $x\in[-1,\ 1]$, $0<\alpha<1$. *It is easy to show that* $\omega(\sigma_\alpha;\ n^{-1})=n^{-\alpha}$, $\omega(\theta_{\sigma_\alpha};n^{-1})=O(n^{\alpha-1})$. *Then Theorem 2 implies*

$$HE([-1,\ 1],\ P_n,\ \sigma_\alpha)=O(n^{-1}).$$

Corollary 2. *For the function*

$$\psi(x)=\begin{cases}(\ln|x|^{-1})^{-1}, & x\in[-e^{-1},\ e^{-1}]=\Delta,\ x\neq 0,\\ 0 & ,\ x=0\end{cases}$$

we have $\omega(\psi;\ n^{-1})=O((\ln n)^{-1})$, $\omega(\theta_\psi;\ n^{-1})=O(\ln^2 n/n)$. *Then Theorem* 2 *implies* $HE(\Delta, P_n;\ \psi)=O(\ln\ln n/n)$.

3. Local Approximation of Function. Using the same method, described above, the following theorem may be proved.

Theorem 3. *For every* 2π*-periodic bounded function* f *there exists a trigonometric polynomial* τ_n *of degree* $\leq n$ *such that for every* x *the inequality* $|f(x)-\tau_n(x)|\leq C_1\omega(f;\ x;\ C_2\ln(e+n\omega(f;\ n^{-1})\omega(\theta_f;\ n^{-1}))/n)+C_3\ln(e+n\omega(f;\ n^{-1})\omega(\theta_f;\ n^{-1}))/n$ *holds true, where* C_1, C_2, C_3 *are absolute constants.*

This theorem generalizes a result of Popov [5], which states that

$$|f(x)-\tau_n(x)|\leq\omega(f;\ x;\ \pi^2 e\ln(n\omega(f;\ n^{-1}))/n)+O(n^{-1}).$$

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