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CONSTRUCTION OF SEMI-GROUPS OF OPERATORS IN $L^2(\Omega)$

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Summary. If $\{T(t): t \geqq 0\}$ is a strongly continuous semi-group of operators in a Banach space X, this semi-group is completely determined by its infinitesimal generator B, defined by

$$Bf = \lim_{t \downarrow 0} (T(t) - I) f / t, \quad f \in D(B) \subset X,$$

where I is the identity operator in X. Moreover, if there is an approximation process $\{A_\lambda : \lambda \geqq 1\}$ in X, converging to the identity operator in X, with the so-called Voronovskaya property concerning the asymptotic behaviour

$$Af = \lim_{\lambda \to \infty} \lambda (A_\lambda - I) f, \quad f \in D(A) \subset X$$

and $B = A$, then the iteration theorem of Trotter gives

$$s - \lim_{\lambda \to \infty} A_\lambda^{[\lambda t]} = T(t),$$

if there is satisfied a number of additional conditions. With concrete approximation processes only in a few cases it has been proved, by means of ad hoc methods, that semi-groups of operators can be constructed by means of iteration. In this paper we prove the possibility of such a construction under fairly general conditions on the operator A in the space $L^2(\Omega)$, $\Omega \subset R^3$, $\Omega = \{x : |x| < 1\}$ and in a certain subspace H of $L^2(\Omega)$.

1. Two Hilbert Spaces in $L^2(\Omega)$. Let Ω be the open unit sphere in $\boldsymbol{R}^3$ with center the origin and let $\partial\Omega$ be its boundary. By $C(\Omega)$ we denote the space of continuous complex-valued functions, defined on Ω, and by $C^k(\Omega)$, $k \in \{1, 2, \ldots; \infty\}$ the subspace of $C(\Omega)$ of functions, which are k-times continuously differentiable on Ω. The continuous extensions to $\overline{\Omega}$ are denoted by $C(\overline{\Omega})$ and $C^k(\overline{\Omega})$. $C_0^k(\Omega)$ is the subspace of functions in $C^k(\Omega)$ with compact support in Ω. Outside Ω all functions are defined by zero.

Let the real function α satisfy the following conditions:

(1.1) (i) $\alpha^{1/2} \in C^\infty(\Omega) \cap C(\overline{\Omega})$, $\quad \alpha \in C^2(\overline{\Omega})$,

(ii) $\alpha(x) > 0, \quad x \in \Omega,$

$\alpha(x) = 0, \quad x \in \Omega,$

and let the real functions b^i $(i = 1, 2, 3)$ satisfy

(1.2) $$b^i \in C^\infty(\Omega) \cap C(\overline{\Omega}) \quad (i = 1, 2, 3).$$

Suppose there exists a real function $l \in C^1(\Omega)$, bounded below and non-decreasing, if $|x|$ is non-decreasing, such that

$$b^i = \alpha D_i l \quad (i=1, 2, 3). \tag{1.3}$$

Let the partial differential expression $A = A(x, D)$ be given by

$$(Af)(x) = A(x, D) f = \alpha(x)(\Delta f)(x) + \sum_{i=1}^{3} b^i(x)(D_i f)(x), \tag{1.4}$$

where α and b^i satisfy the conditions (1.1)—(1.3). Thus A degenerates at the boundary $\partial\Omega$. We define the real function on $\overline{\Omega}$ by

$$\gamma(x) = \alpha(x) \exp(-l(x)).$$

It is clear that

(i) $\gamma \in C^\infty(\Omega) \cap C(\overline{\Omega})$,

(ii) $\gamma(x) > 0,\ x \in \Omega$,

$\gamma(x) = 0,\ x \in \partial\Omega$.

In the linear space $C_0^\infty(\Omega)$ we define a scalar product $(\cdot, \cdot)_H$ by means of

$$(f, g)_H = \int_\Omega f \bar{g} \frac{dx}{\gamma}.$$

This scalar product generates a norm $\|\cdot\|_H$, defined by

$$\|f\|_H^2 = \int_\Omega |f|^2 \frac{dx}{\gamma}.$$

Now we define the Hilbert space H by

Definition 1. The Hilbert space H is the completion of $C_0^\infty(\Omega)$ with respect to the scalar product $(\cdot, \cdot)_H$.

We define a second scalar product $(\cdot, \cdot)_K$ in $C_0^\infty(\Omega)$ by means of

$$(f, g)_K = (f, g)_H + \sum_{i=1}^{3} (\alpha^{1/2} D_i f, \alpha^{1/2} D_i g)_H.$$

The norm $\|\cdot\|_K$, corresponding to this scalar product, is defined by

$$\|f\|_K^2 = \|f\|_H^2 + \|\alpha^{1/2} D_i f\|_H^2.$$

We define the Hilbert space K by

Definition 2. The Hilbert space K is the completion of $C_0^\infty(\Omega)$ with respect to the scalar product $(\cdot, \cdot)_K$.

It is easily verified that

$$C_0^\infty(\Omega) \subset K \subset H \subset L^2(\Omega), \tag{1.5}$$

$$\|f\|_H \leq \|f\|_K, \quad \|f\|_{L^2(\Omega)} \leq \sup_{x \in \Omega} \gamma(x) \|f\|_H,$$

K is dense in H, H is dense in $L^2(\Omega)$, H and K are separable.

In order to characterize all elements $f \in H$, we have as a corollary of (1.5)

Theorem 1. *H is isomorphic to a subspace of the space of all distributions in* Ω.

By this theorem (distributional) differentiation of elements of H is meaningful. If $f \in H$ and if α is a multi-index, then $D^\alpha f$ is defined by $(D^\alpha f)(\varphi) = (-1)^{|\alpha|} f(D^\alpha \varphi)$. Let $I: K \to H$ be the natural embedding operator and $I^*: H \to K$ the adjoint operator defined by

$$(If, g)_H = (f, I^* g)_K, \quad f \in K, \; g \in H,$$

then we have the following characterization theorem for the range $R(I^*)$ of I^*.

Theorem 2. $I^*: H \to K$ *is an injection with the range*

$$R(I^*) = \{k \in K : |(f, k)_K| \leq K_K \|f\|_H, \; f \in K\}. \tag{1.6}$$

Proof. If $g \in H$, and $I^* g = 0 \in K$, then $(If, g)_H = 0$ for all $f \in K$. Since the range $R(I)$ of I is dense in H, we have $g = 0$. So I^* is an injection.

If $k \in R(I^*)$, then there exists a $h \in H$, such that $I^* h = k$ and it follows that

$$|(f, k)_K| \leq K_K \|f\|_H, \; f \in K \tag{1.7}$$

where the constant K_K is independent of f.

Conversely, if the linear functional on $K: f \to (f, k)_K$ with $k \in K$, satisfies relation (1.7), then this functional is bounded in the topology of H. It follows from the Riesz representation theorem that there exists an element $h \in H$ such that $(If, h)_H = (f, k)_K$ and clearly $k = I^* h \in R(I^*)$. This proves (1.6).

2. The Approximation Process $\{A_\lambda \mid \lambda \geq 1\}$ in $L^2(\Omega)$ and in H. Let M and L be the cubes in $\boldsymbol{R}^3$ with edges 1 defined by

$$M = \{x : \max_{i=1,2,3} |x_i| \leq 1/2\}$$

$$L = \{x : -1 \leq x_i \leq 0, \quad i = 1, 2, 3\}.$$

We define the characteristic function χ_V with respect to a set $V \subset \boldsymbol{R}^3$ by $\chi_V(x) = 1$ if $x \in V$, $\chi_V(x) = 0$ if $x \notin V$. In $L^2(\Omega)$, the space of complex-valued functions, which are square-integrable over Ω, we define the operators S_λ, T_λ and A_λ by

$$(S_\lambda f)(x) = \int_{\boldsymbol{R}^3} f(x - 4\sqrt{3\alpha(x)}\, t)\, \lambda^3 \chi_M(\lambda t)\, dt,$$

$$(T_\lambda f)(x) = \int_{\boldsymbol{R}^3} f(x_1 - 4b^1(x) t_1, x_2 - 4b^2(x) t_2, x_3 - 4b^3(x) t_3)$$

$$\times \lambda^3 \chi_L(\lambda t_1, \lambda t_2, \lambda t_3)\, dt_1\, dt_2\, dt_3,$$

$$(A_\lambda f)(x) = 2^{-1}((S_{\sqrt{\lambda}} f)(x) + (T_\lambda f)(x)) \tag{2.1}$$

for $x \in \Omega, \lambda \geq 1$. Except for some small modifications we studied these operators in [4]. In that paper we proved the following

Theorem 3. *Let the expression A_λ be defined by* (2.1). *Then for λ sufficiently large A_λ defines an operator $L^2(\Omega) \to L^2(\Omega)$ with the following properties:*

(a) *A_λ is strongly convergent to the identity operator on $L^2(\Omega)$.*

(b) $\|A_\lambda\| = 1 + O(1/\lambda), \quad \lambda \to \infty$.

(c) *If $A: D(A) \to L^2(\Omega)$ is defined by*

$D(A) = (C^2(\Omega), \|\cdot\|_{L^2(\Omega)})$,

$$(2.2)\qquad (A f)(x)=\alpha(x)(\Delta f)(x)+\sum_{i=1}^{3} b^i(x)(D_i f)(x),$$

then we have $\lim_{\lambda\to\infty}\lambda(A_\lambda f-f)=Af$ *in the topology of* $L^2(\Omega)$.

In view of the theorem of Trotter we prove the following

Theorem 4. *The operator* $A:D(A)\to L^2(\Omega)$ *is closable.*

Proof. A is closable in $L^2(\Omega)$, if there exists a linear extension of A, which is closed in $L^2(\Omega)$. This is equivalent with the statement, that $\lim_{n\to\infty}\|f_n\|_{L^2(\Omega)}=0$ and $\lim_{n\to\infty}\|Af_n-g\|_{L^2(\Omega)}=0$, $g\in L^2(\Omega)$, implies $g=0$. We denote the formal transpose of A by ${}^tA$. If we consider (Af_n) as a sequence of distributions, we have for each $\varphi\in C_0^\infty(\Omega)$

$$|\langle f_n, {}^tA\varphi\rangle-\langle g,\varphi\rangle|=|\langle Af_n-g,\varphi\rangle|$$

$$=|\int_\Omega (Af_n-g)(x)\varphi(x)\,dx|\le\|Af_n-g\|_{L^2(\Omega)}\cdot\|\varphi\|_{L^2(\Omega)}.$$

The left-hand side tends to $|\langle g,\varphi\rangle|$ and the right-hane side tends to zero, thus $\langle g,\varphi\rangle=0$ for each $\varphi\in C_0^\infty(\Omega)$. This implies $g=0$. □

The space H is a weighted $L^2(\Omega)$-space. If we consider the restrictions $\tilde{A}_\lambda:H\to H$ of A_λ and if we write A_λ, too, for $\tilde{A}_\lambda$, then we can prove, using the fact that $\gamma(x-\alpha(x)t/\lambda)/\gamma(x)$ is bounded on Ω, uniformly in t and λ, if λ is sufficiently large.

Theorems 3′ and 4′. *The Theorems 3 and 4 remain valid, if we change* $L^2(\Omega)$ *into* H.

3. The Range of the Operator $1-A$. Let the partial differential operator $A\equiv A(x,D)$ from H into H with natural domain $D(A)$ be defined by the expression (2.2). We denote the identity operator on H or on $L^2(\Omega)$ by the number 1. Moreover, let $A_1=1-A:D(A)\to H$. Then clearly $C_0^\infty(\Omega)\subset D(A)$. The next lemma on A_1 is of crucial importance.

Lemma 5. *For each* $f\in C_0^\infty(\Omega)$ *we have* $A_1 f=(I^*)^{-1}f$.

Proof. Let $f\in C_0^\infty(\Omega)$. For each $\varphi\in C_0^\infty(\Omega)$ we have by partial integration

$$(A_1 f,\varphi)_H=(f,\varphi)_H-(Af,\varphi)_H$$

$$=(f,\varphi)_H+\sum_{i=1}^{3}(\alpha^{1/2}D_i f,\alpha^{1/2}D_i\varphi)_H=(f,\varphi)_K=((I^*)^{-1}f,\varphi)_H.$$

Since $C_0^\infty(\Omega)$ is dense in H, we see that $A_1 f=(I^*)^{-1}f$.

Theorem 6. *The range* $R(A_1)$ *of* A_1 *is dense in* $L^2(\Omega)$ *and in* H.

Proof. Since H is a dense subspace of $L^2(\Omega)$, it is sufficient to prove that $R(A_1)$ is dense in H. Therefore let $f\in H$ and let $g=I^*f$ belong to $R(I^*)\subset K$. There exists a Cauchy sequence (g_n) in $C_0^\infty(\Omega)\cap K$, converging to g in K. Consider the sequence (f_n) in $C_0^\infty(\Omega)\cap H$ with $f_n=A_1 g_n$. By Lemma 5 we have for all $\varphi\in C_0^\infty(\Omega)$

$$(f_n-f,\varphi)_H=(A_1 g_n-(I^*)^{-1}g,\varphi)_H=((I^*)^{-1}(g_n-g),\varphi)_H=(g_n-g,\varphi)_K.$$

Thus $\lim_{n\to\infty}(f_n-f,\varphi)_H=0$ for all $\varphi\in C_0^\infty(\Omega)$. The norm of the functional on $H:\varphi\to(f_n-f,\varphi)_H$ is equal to $\|f_n-f\|_H$. The theorem of Banach-Steinhaus gives $\lim_{n\to\infty}\|f_n-f\|_H=0$. Since $(f_n)\subset R(A_1)$, the theorem is proved now.

4. The Construction of a Semi-Group. For the construction of the semigroup with the operator A, defined by the expression (2.2) as infinitesimal generator, we use the theorem of Trotter [9], about in the version of [2].

Theorem 7. *Let $\{A_\lambda : \lambda \geq 1\}$ be an approximation process in a Banach space X, A a closed operator with domain $D(A) \subset X$ and range $R(A) \subset X$ and let there exists a function $\varphi : [1, \infty) \to (0, \infty)$, such that there are satisfied the following conditions:*

(a) $\| A_\lambda^n f \|_X \leq M \exp (Kn/\varphi(\lambda)) \| f \|_X$,

where M and K are constants independent of f and n.

(b) $\lim_{\lambda\to\infty} \varphi(\lambda) (A_\lambda - 1) f = Af$, $f \in D(A)$ *(Voronovskaya-property).*

(c) *$D(A)$ is dense in X.*

(d) *There exists a positive number μ, such that the range $R(\mu - A)$ of $\mu - A$ is dense in X.*

Then the operator A is the infinitesimal generator of a sémi-group $\{T(t) : t \geq 0\}$ of opérators in X of class C_0 and for each $f \in X$ we have

$$\lim_{\lambda\to\infty} A_\lambda^{[\varphi(\lambda)t]} f = T(t) f, \tag{4.1}$$

uniformly on each bounded t-interval. □

With the support of the Hilbert spaces H and K it is easily verified that our approximation process $\{A_\lambda : \lambda \geq 1\}$ of Section 2 with $X = H$ or $L^2(\Omega)$, $\varphi(\lambda) = \lambda$ and as A the closed extension of (2.2) satisfies the conditions of the theorem of Trotter. For (a) follows from Theorem 3(b), (b) from Theorem 3(c) and Theorem 4(c) from (1.5) and (d) from Theorem 6 with $\mu = 1$. Thus (4.1) holds, where $\{T(t)\}$ is a strongly continuous semi-group of operators in $L^2(\Omega)$ with A as infinitesimal generator.

5. Application. In Section 2 we introduced an approximation process $\{A_\lambda\}$, converging to the identity in H, satisfying the conditions of Trotter and with the Voronovskaya property (2.2). This approximation process can serve as an application of our theory. We take $a(x) = (1 - |x|^2)^q$ and $b^i(x) = 2x_i \times (1 - |x|^2)^{q-1}$, $q \geq 2$ $(i = 1, 2, 3)$. It is easy to verify that $l(x) = -\ln(1 - |x|^2)$. Then $\gamma(x) = (1 - |x|^2)^{q+1}$, and H and K can be defined as in Section 1.

As a conclusion we have: if $q \geq 2$, then iteration of this approximation process leads to a semi-group of operators in H, as well as in $L^2(\Omega)$, with the operator

$$(1 - |x|^2)^q \sum_{i=1}^{3} D_i^2 + 2(1 - |x|^2)^{q-1} \sum_{i=1}^{3} D_i$$

as infinitesimal generator.

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