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ON THE APPROXIMATION OF HOLOMORPHIC FUNCTIONS IN THE UNIT BALL OF C^n

V. Totik

Summary. In connection with a result of Rudin [3] and Ahern and Schneider [1] we investigate the approximation of holomorphic functions in the unit ball of C^n by polynomials. The analogues of classical direct and indirect results are proved. We give also an example, the medhod of which can be extended to many other cases to prove that the results are best possible. Finally, we point out to the fact that the integral case can be treated similarly. Our method is elementary and is based on well-known approximation results.

1. Introduction. Rudin [3] proved, among others, that, if the functions f_ζ (for the notations see below) constitute a bounded set in Lip α $(0<\alpha<1)$ and f is holomorphic in B, then f itself belongs to Lip α in $\bar{B}$. Ahern and Schneider [1] extended this result by showing that, if the assumption 'f is holomorphic' is dropped, then the conclusion still remains true for $C[f]$ (actually Ahern and Schneider considered more general domains than the ball B). In this simple note we prove that these results as well as considerable generalizations of them can be obtained by using well-known approximation results and arguments. Since the proofs or at least part of them run on standard lines, we can be very short and point out only to the key steps of the proofs.

2. Notations. We use the notations of [2]. Let C^n be the cartesian product of n copies of C. We introduce the inner product $\langle z, w\rangle=\Sigma_{j=1}^n z_j\bar{w}_j$ $(z, w\in C^n)$ and the norm $|z|=\langle z, z\rangle^{1/2}=\sqrt{\Sigma_{j=1}^n |z_j|^2}$. $e_1, \dots, e_n$ denote the standard orthonormal basis in C^n, where e_k is the ordered n-tuple that has 1 in the k-th spot and 0 everywhere else.

$B=B_n=\{z\in C^n: |z|<1\}$ is the open unit ball in C^n; $S=S_n=\partial B$ its boundary is the set of all unit vectors $|z|=1$. We shall omit the subscript in B_n, if the dimension will be clear. We set $\bar{B}=B\cup S$. U stands for B_1, i. e. for the open unit disc in C. We let σ be the rotation-invariant positive Borel measure on S, for which $\sigma(S)=1$.

A function $f: B\to C$ is said to be holomorphic, if it is continuous and it is holomorphic in each variable z_k separately. The set of holomorphic functions $f: B\to C$ will be denoted by $H(B)$. $A(B)=H(B)\cap C(\bar{B})$ is the ball algebra, i. e. the set of those $f: \bar{B} \to C$, which are continuous on $\bar{B}$ and which are holomorphic in B.

The term multi-index refers to an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers α_i. We put $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. Every $\in H(B)$ has an expansion $f(z) = \Sigma_\alpha c_\alpha z^\alpha$, valid for all $z \in B$. If we collect here those α's, for which $|\alpha|$ is the same, we get $f(z) = \Sigma_{k=0}^\infty \Sigma_{|\alpha|=k} c_\alpha z^\alpha = \Sigma_{k=0}^\infty F_k(z)$, the so-called homogeneous expansion of f. The radial derivatfive $\mathcal{R} f$ of f is defined as $\mathcal{R} f(z) = \Sigma_{k=0}^\infty k F_k(z)$. If f is a polynomial of n variables, its degree is the largest k, for which we have $F_k(z) \neq 0$.

If $f \in H(B)$, $\zeta \in S$, let $f_\zeta(\lambda) = f(\zeta\lambda)\,(\lambda \in U)$. These so-called slice functions belong to $H(U)$. Clearly, $(\mathcal{R} f)_\zeta(\lambda) = \lambda f'_\zeta(\lambda)$.

The Cauchy kernel for B is the function $C(z, \zeta) = (1 - \langle z, \zeta \rangle)^{-n}$, defined for all $\langle z, \zeta \rangle \in C^n \times C^n$ with $\langle z, \zeta \rangle \neq 1$. If $f \in L^1(\sigma)$, define its Cauchy integral as $C[f](z) = \int_S C(z, \zeta) f(\zeta) d\sigma(\zeta)$. If $f \in A(B)$, then $C[f](z) = f(z)$ for $z \in B$ (see [2, p. 39]). Using Proposition 1.4.7. (1), from [2] we have also

$$C[f](z) = \int_S d\,\sigma(\zeta)\,(2\pi)^{-1} \int_0^{2\pi} f(\zeta e^{i\vartheta})\,(1 - \langle z, \zeta e^{i\vartheta} \rangle)^{-n}\, d\vartheta \quad (z \in B).$$

Finally, let us agree that K denotes always a positive constant, not necessarily the same at each occurrence, which may depend on a parameter, written as a subscript (K_r), and also on the dimension n, but independent of the other quantities.

3. Moduli of Smoothness and Approximation. In this paragraph we prove the analogues of some well-known results for the case $f \in H(B)$. Let $E_n(f) = \inf_{P_n} \sup_{z \in B} |f(z) - P_n(z)|$, where the infimum is taken over all polynomials of degree at most n.

Although we do not need it, we mention that a simple compactness argument gives that in the definition of $E_n(f)$ the inf is attained for some polynomial P_n. If $z \in B$, $z + h \in B$ $(h \in C^n)$, let $\Delta_h^1(f; z) = f(z+h) - f(z)$, and we define inductively $\Delta_h^r(f; z)$, as $\Delta_h^{r-1}(f; z+h) - \Delta_h^{r-1}(f; z)$. The r-th modulus of smoothness of f is defined by $\omega_r(f; \delta) = \sup\{|\Delta_h^r(f; z)| : z, z+hr \in B, |h| \leq \delta\}$. Especially, $\omega_1(f; \delta)$ is the modulus of continuity of f. A number of classical approximation results treat the relation between the best uniform approximation of a function (or of a derivative of it) and its moduli of smoothness. Now we shall carry over some of these theorems to functions, defined in B.

Let us begin with the simple

Lemma 1. *If f is a polynomial of n variable and of degree m and $\mathcal{R} f$ denotes its radial derivative, then $\|\mathcal{R} f\| \leq m \|f\|$.*

Here and in the following $\|g\|$ denotes $\sup\{|g(z)| : z \in B\}$.

Proof. Since $(\mathcal{R} f)(\lambda\zeta) = \lambda f'_\zeta(\lambda)$ $(\zeta \in S, \lambda \in U)$ (see [2, p. 103]), the assertion follows from the maximum modulus principle and from Bernstein's inequality: $|f'_\zeta(\xi)| \leq m \max\{|f_\zeta(\xi)| : \xi \in \partial U\}$ $(\xi \in \partial U)$.

Assertion 1. If $f \in H(B)$, then for $k = 1, 2, \ldots$

$$E_m(\mathcal{R}^k f) \leq K_k\Big(m^k E_m(f) + \sum_{\nu=m+1}^{\infty} \nu^{k-1} E_\nu(f)\Big) \quad (m = 1, 2, \ldots).$$

Proof. Let P_m be a polynomial of order at most m with $\|f-P_m\| \leq 2E_m(f)$. We may suppose $\sum_{\nu=1}^{\infty} \nu^{k-1}E_\nu(f)<\infty$ and in this case we have by Lemma 1 $\mathscr{R}^k f=\mathscr{R}^k P_m+\sum_{i=0}^{\infty} \mathscr{R}^k(P_{2^{i+1}m}-P_{2^i m})$.

Since $\mathscr{R}^k P_m$ is a polynomial of order at most m and since $\|P_{2^{i+1}m} - P_{2^i m}\| \leq \|P_{2^{i+1}m}-f\|+\|P_{2^i m}-f\| \leq 4E_{2^i m}(f)$, the assertion follows by a repeated application of Lemma 1:

$$E_m(\mathscr{R}^k f)\leq K\sum_{i=0}^{\infty}(2^{i+1}m)^k E_{2^i m}(f)\leq K(m^k E_m(f)+\sum_{\nu=m+1}^{\infty}\nu^{k-1}E_\nu(f)).$$

Assertion 2. If $f\in H(\boldsymbol{B})$ and $k\geq 1$, then

(1)
$$E_{2m}(f)\leq K_k\, m^{-k}E_m(\mathscr{R}^k f).$$

Proof. Let for $0<r<1$ $g(r;z)=g(z)=f(rz)$. Clearly

(2)
$$E_m(\mathscr{R}^k g(r;\cdot))\leq E_m(\mathscr{R}^k f)$$

is satisfied (use that $(\mathscr{R}^k g)(z)=(\mathscr{R}^k f)(rz)$). First we prove (1) for g.

Since g is C^∞ on $\bar{\boldsymbol{B}}$, the differentiations below are justified.

Let $E_m^T(h)$ be the best uniform approximation of the 2π-periodic function $h:\boldsymbol{R}\to\boldsymbol{C}$ by trigonometric polynomials of order at most m. Let us consider the function $g_\zeta(e^{i\vartheta})$ for fixed $\zeta\in\boldsymbol{S}$.

Since $dg_\zeta(e^{i\vartheta})/d\vartheta=i(\mathscr{R}g)_\zeta(e^{i\vartheta})$, we get by the well-known inequality $E_m^T(h)\leq m^{-1}K\,E_m^T(h')$ of trigonometric approximation that

$$E_m^T(g_\zeta(e^{i\,\cdot}))\leq m^{-1}K\,E_m^T((\mathscr{R}g)_\zeta(e^{i\,\cdot})).$$

Iterating this k times and using Assertion 5 below, we obtain (1) for g. This and (2) give

(3)
$$E_{2m}(g(r;\ \cdot))\leq K_k m^{-k}E_m(\mathscr{R}^k f).$$

Now, if the right-hand side of (3) is infinite, then (1) is satisfied vaguely and in the opposite case we get (1) from (3) by a standard compactness argument (letting $r\to 1$).

Before turning to the relations between $E_m(f)$ and $\omega_r(f;\delta)$, let us prove

Assertion 3. If $f\in H(\boldsymbol{B})$, then $\sup_{\zeta\in\boldsymbol{S}} E_{2m}(f_\zeta)\leq E_{2m}(f)\leq 4\sup_{\zeta\in\boldsymbol{S}}E_m(f_\zeta)$ $(m=1,2,\dots)$.

Since f_ζ is of one variable, this assertion allows us to consider only the one-dimensional case in certain cases.

Proof. Let $f(z)=\sum_{k=0}^{\infty}F_k(z)$ be the homogeneous expansion of f, $S_\nu(f;z) =\sum_{k=0}^{\nu}F_k(z)$, and $\tau_k(f;z)=k^{-1}\sum_{\nu=k+1}^{2k}S_\nu(f;z)$ the de la Vallée Poussin means of $\{S_\nu\}$. Since $E_{2m}(f)\leq\|f-\tau_m(f)\|$, it is enough to show for proving the right inequality of Assertion 3, that $\|f-\tau_m(f)\|\leq 4\sup_{\zeta\in S}E_m(f_\zeta)$ and to this end it is sufficient that

$$\|(f-\tau_m(f))_\zeta\|\leq 4E_m(f_\zeta)\quad(\zeta\in\boldsymbol{S})$$

be satisfied. However, $\tau_m(f)_\zeta=\tau_m(f_\zeta)$ and thus, using the inequality ([8, p. 115])

$$|\tau_m^*(h)-h|\leq 4E_m^T(h),\quad (h\in C_{2\pi})$$

where $\tau_m^*(h)$ denotes the de la Vallée Poussin means of the Fourier series of h, we get by the maximum modulus principle

$$\text{(4)}\qquad \|(f-\tau_m(f))_\zeta\| = \sup_{0<r<1} \|f_\zeta(re^{i\cdot})-\tau_m(f)_\zeta(re^{i\cdot})\|_{c_{2\pi}}$$

$$= \sup_{0<r<1} \|f_\zeta(re^{i\cdot})-\tau_m^*(f_\zeta(re^{i\cdot}))\|_{c_{2\pi}}$$

$$\leq \sup_{0<r<1} 4E_m^T(f_\zeta(re^{i\cdot}))\leq 4E_m(f_\zeta).$$

The left inequality in Assertion 3 is trivial.

Lemma 2. *If f is a polynomial of one variable and of degree m, then*

$$\sup\{|f(z)|: z\in \boldsymbol{U}\}\leq 2\sup\{|f(z)|:|z|\leq 1-1/2m\}.$$

Proof. Let $M=\sup_{z\in \boldsymbol{U}}|f(z)|=|f(z_0)|$, where $z_0\in\partial\boldsymbol{U}$. Bernstein's inequality for the derivative of a trigonometric polynomial together with the maximum modulus principle give that $|f'(z)|\leq mM$ for $z\in\boldsymbol{U}$ and thus

$$|f(z_0)-f(z_0(1-1/2m))| = \left|\int_{z_0(1-1/2m)}^{z_0} f'(\tau)\,d\tau\right|\leq M/2,$$

which proves the lemma.

After this we can prove

Theorem 1. *If $f\in H(\boldsymbol{B})$, then $E_m(f)\leq K_r\,\omega_r(f;1/m)$ $(r,\ m=1,2,\ldots)$.*

Proof. By Assertion 3 we have to consider only the case $n=1$ (take into account that $\omega_r(f_\zeta;\ 1/m)\leq\omega_r(f;\ 1/m)$ for every $\zeta\in\boldsymbol{S}$). We may suppose also that $m\geq r$.

First we prove that for $|z|\leq 1-1/2m$

$$\text{(5)}\qquad |f^{(r)}(z)|\leq K_r m^r\omega_r(f;\ 1/m).$$

In fact, by Cauchy formula $f^{(r)}(z)=(r!/2\pi)\int_0^{2\pi} f(z+k\tau e^{it})(k\tau)^{-r}e^{-irt}dt$, provided $|k\tau|\leq 1-|z|$. Let here $\tau=1/2mr$. Multiplying the previous equality by $(-1)^{r+k}(k\tau)^r\binom{r}{k}$, adding these for $k=1,2,\ldots,r$ to the equation $0=(-1)^r(r!/2\pi)\int_0^{2\pi} f(z)e^{-irt}dt$ and taking into account that

$$(-1)^r\sum_{k=1}^{r}(-1)^k\binom{r}{k}k^r=\Delta_1^r(x^r,\ 0)=r!,$$

we obtain

$$|f^{(r)}(z)| = \left|(1/2\pi)\int_0^{2\pi}\Delta^r_{\tau e^{it}}(f;\ z)\tau^{-r}e^{-irt}\,dt\right|\leq\omega_r(f;\tau)\tau^{-r}$$

$$\leq K_r m^r\omega_r(f;\ 1/m).$$

By (5) $f^{(r)}(z)$ can be approximated in the disk $\{|z|\leq 1-1/2m\}$ by the zero polynomial in order $m^r\omega_r(f;\ 1/m)$, which in turn implies (see the proof of Assertion 2), that

(6) $$|f(z)-P_m(z)|\leq K_r\,\omega_r(f;\,1/m)\quad(|z|\leq 1-1/2m)$$

for some polynomial $P_m(z)$ of order at most m. To prove (6) for all $z\in \boldsymbol{U}$, let us consider the polynomial $g(z)=\Delta^r_{z/m}(P_m;\,z-rz/m)$. If $|z|\leq 1-1/2m$, then (6) gives

$$|g(z)|\leq|\Delta^r_{z/m}(P_m-f;\,z-rz/m)|+|\Delta^r_{z/m}(f;\,z-rz/m)|\leq K\omega_r(f;\,1/m)$$

and thus, according to Lemma 2, this inequality is valid for all $z\in\boldsymbol{U}$. Now for arbitrary $z\in\boldsymbol{U}$ we get

$$|f(z)-P_m(z)|\leq|g(z)|+|\Delta^r_{z/m}(f;\,z-rz/m)|$$

$$+\left|\sum_{k=1}^{r}(-1)^k\binom{r}{k}(f-P_m)(z-kz/m)\right|\leq K\omega_r(f;\,1/m),$$

since for $r\geq k\geq 1$ $|z-kz/m|\leq 1-1/2m$.

Lemma 3. *If f is a polynomial of n variable and of order m, then* $\|\operatorname{grad} f\|_{C^n}\leq Km\,\|f\|$. *Here* $\operatorname{grad} f=(\partial f/\partial z_1,\,\ldots,\,\partial f/\partial z_n)$.

Proof. Applying Lemma 2 to the polynomials $(\partial f/\partial z_k)_\zeta$ $(\zeta\in\boldsymbol{S})$, it is enough to prove $|\partial f/\partial z_k(z)|\leq Km\,\|f\|$ for $k=1,2,\ldots,n$ and $|z|\leq 1-1/2m$. For such z's the Cauchy formula gives

$$|\partial f/\partial z_k(z)|=\left|(2\pi)^{-1}\int_0^{2\pi} f(z+e_ke^{it}/2m)(e^{it}/2m)^{-1}dt\right|\leq 2m\,\|f\|$$

and we are over.

Next we estimate the moduli of smoothness by the quantities $E_m(f)$:

Assertion 4. If $f\in H(\boldsymbol{B})$, then $\omega_r(f;1/m)\leq K_r\,m^{-r}\sum_{\nu=1}^{m}\nu^{r-1}E_\nu(f)$ $(r,\,m=1,\,2,\ldots)$.

Using Lemma 3, the proof if standard (see e. g. [4, p. 333]).

4. Boundary Values and Approximation. Now we turn to the investigation of $E_m(f)$ through the properties of the boundary values of f.

Let $f\in H(\boldsymbol{B})$ be bounded, i. e. $f\in H^\infty(\boldsymbol{B})$ (only for such functions the approximation question is of interest). At almost all $\zeta\in\boldsymbol{S}$ f has the so-called radial limit

$$f^*(\zeta)=\lim_{r\nearrow 1} f(r\zeta)$$

(see [2, p. 85]). Thus we can extend the domain of f to $\boldsymbol{S}$ by putting $f(\zeta)=f^*(\zeta)$ for $\zeta\in\boldsymbol{S}$.

Let for $f:\boldsymbol{S}\to\boldsymbol{C}$ $E^\zeta_m(f)=E^T_m(f(\zeta e^{i\,\cdot}))$, $\omega^\zeta_r(f;\,\delta)=\omega_r(f(\zeta e^{i\,\cdot});\,\delta)$ and

$$E^*_m(f)=\sup_{\zeta\in\boldsymbol{S}}E^\zeta_m(f);\quad \omega^*_r(f;\,\delta)=\sup_{\zeta\in\boldsymbol{S}}\omega^\zeta_r(f;\,\delta).$$

Note that these are concerned only to the slice functions $f(\zeta e^{i\vartheta})$, the behaviour of f, perpendicular to the circles $\zeta e^{i\vartheta}(\zeta\in\boldsymbol{S},\ \vartheta\in\boldsymbol{R})$, is not taken into account.

Our first result is

Assertion 5. For $f \in H^\infty(\boldsymbol{B})$ we have

$$E_{2m}(f) \leq 4E^*_m(f) \quad (m=1, 2, 3, \dots) \tag{7}$$

The proof is implicit in the proof of Assertion 3 (take into account that

$$|f_\zeta(e^{i\vartheta}) - \tau^*_m(f_\zeta(e^{i\cdot}); \vartheta)| \,(=|f_\zeta(e^{i\vartheta}) - \tau_m(f_\zeta; e^{i\vartheta})|) \leq 4E^\zeta_m(f) \quad (\vartheta \in \boldsymbol{R})$$

implies $\| f_\zeta(re^{i\cdot}) - \tau^*_m(f_\zeta(re^{i\cdot}))\|_{C_{2\pi}} \leq 4E^\zeta_m(f)$ $(0<r<1)$ and see then formula (4))·

Corollary 1. *For* $f \in H^\infty(\boldsymbol{B})$ *and* $r \geq 1$ $E_m(f) \leq K\omega^*_r(f; 1/m)$ $(m=1, 2, \dots)$

This follows from (7) and from $E^\zeta_m(f) \leq K\omega^\zeta_r(f; 1/m)$ (see [4, p. 262])

Another interesting consequence is

Corollary 2. *If* $f \in H^\infty(\boldsymbol{B})$, *then*

a) $\omega_s(f; \delta) \leq K_{r,s}\delta^s \int_\delta^{2\pi} t^{-s-1}\omega^*_r(f; t)dt,$

b) α) $\omega^*_1(f; \delta) \leq K\omega_1(f; \delta),$

β) $\omega^*_s(f; \delta) \leq K_{r,s}\,\delta^s \int_\delta^{2\pi} t^{-s-1}\,\omega_r(f; t)\,dt \quad (r, s \geq 1).$

The point is that we can compare the smoothness of f on lines ($\omega_r(f; \delta)$) to that of it on the circles ζe^{it}($\omega^*_r(f; \delta)$).

Proof. a) follows from Corollary 1 and from Assertion 4. b) α·, is trivial, and b) β·, follows from Theorem 1 and from the analogue of Assertion 4 for trigonometric approximation (see [4, p. 333]).

After this we consider the Cauchy-integral of functions $f \in L^1(\sigma)$.

Lemma 4. *If* $f \in L^1(\sigma)$, $f^\zeta(\vartheta) = f(\zeta e^{i\vartheta})$ $(\zeta \in \boldsymbol{S}, \vartheta \in \boldsymbol{R})$ *is a trigonometric polynomial of order at most m for all* $\zeta \in \boldsymbol{S}$ *and* $\omega^*_1(f; \delta) \leq M\delta$, *then* $\omega^*_2(C[f]; \delta) \leq KmM\delta^2$.

Proof. Let $g = C[f]$. The definition of $C[f]$ and the equality

$$\int_{\boldsymbol{S}} f(\zeta)\,\bar{\zeta}^\alpha d\sigma(\zeta) = \int_{\boldsymbol{S}} d\sigma(\zeta) \frac{1}{2\pi}\int_0^{2\pi} f(\zeta e^{i\vartheta})\, e^{-i|\alpha|\vartheta}\,\bar{\zeta}^\alpha d\vartheta$$

(see [2, p. 15]) give easily that g is a polynomial of order at most m. By a unitary change of variables we have to estimate only $g(e^{it}\boldsymbol{e}_1) - 2g(\boldsymbol{e}_1) + g(e^{-it}\boldsymbol{e}_1)$. Let $a = 1 - 1/2m$ and let us consider

$$|g(ae^{it}\boldsymbol{e}_1) - 2g(a\boldsymbol{e}_1) + g(ae^{-it}\boldsymbol{e}_1)|$$

$$= |\int_{\boldsymbol{S}} d\sigma(\zeta)\, \frac{1}{2\pi}\int_0^{2\pi} f(\zeta e^{i\vartheta})\,((1 - ae^{i(t-\vartheta)}\bar{\zeta}_1)^{-n} - 2(1 - ae^{-\vartheta}\bar{\zeta}_1)^{-n}$$

$$+ (1 - ae^{i(-t-\vartheta)}\bar{\zeta}_1)^{-n})d\vartheta| = |\int_{\boldsymbol{S}} d\sigma(\zeta)\,(2\pi)^{-1}\int_0^{2\pi} (f(\zeta e^{i(\vartheta+t)}) - f(\zeta e^{i\vartheta}))((1 - ae^{-i\vartheta}\bar{\zeta}_1)^{-n}$$

$$-(1 - ae^{-i(\vartheta+t)}\bar{\zeta}_1)^{-n})d\vartheta| \leq \int_{\boldsymbol{S}} d\sigma(\zeta) Mt(2\pi)^{-1}\int_0^{2\pi} |(1 - ae^{-i\vartheta}\bar{\zeta}_1)^{-n} - (1 - ae^{-i(\vartheta+t)}\bar{\zeta}_1)^{-n}|\,d\vartheta$$

$$= Mt\int_{\boldsymbol{S}} |(1 - a\langle \boldsymbol{e}_1; \zeta\rangle)^{-n} - (1 - ae^{-it}\langle \boldsymbol{e}_1; \zeta\rangle)^{-n}|\,d\sigma(\zeta)$$

which is equal by [2, 1.4.5 (2)] to

$$Mt\pi^{-1}(n-1)\iint\limits_{U}(1-r^2)^{n-2}|(1-are^{i\vartheta})^{-n}-(1-ae^{-it}re^{i\vartheta})^{-n}|r\,dr d\vartheta$$

$$=\pi^{-1}Mt(n-1)\iint\limits_{U}(1-r^2)^{n-2}|\int\limits_{-t}^{0}-nare^{i(\vartheta+s)}i(1-are^{i(\vartheta+s)})^{-(n+1)}ds|\,r\,dr d\vartheta$$

$$\leq KMt|\int\limits_{-t}^{0}ds\int\limits_{0}^{1}dr(1-r^2)^{n-2}r^2\int\limits_{0}^{2\pi}|1-are^{i(\vartheta+s)}|^{-(n+1)}d\vartheta|$$

$$\overset{*}{\leq}KMt\,|\int\limits_{-t}^{0}ds\int\limits_{0}^{1}(1-ar)^{-n}(1-r^2)^{n-2}\,dr|\leq KMt(t/(1-a))\leq KMt^2m,$$

where at the $*$ inequality we used that $\int_0^{2\pi}|1-ze^{-is}|^{-n-s}ds\approx(1-|z|)^{-n}$ $z\in\boldsymbol{U})$ (see [2, 1.4.10). What we have proved sofar implies also (use a unitary change of vjariable) that $|g(ae^{i(\vartheta+t)}\boldsymbol{e}_1)-2g(ae^{i\vartheta}\boldsymbol{e}_1)+g(ae^{i(\vartheta-t)}\boldsymbol{e}_1)|$ $\leq KMmt^2$ and thus, applying Lemma 2 to the polynomial $g(\lambda e^{it}e_1)-2g(\lambda\boldsymbol{e}_1)$ $+g(\lambda e^{-it}\boldsymbol{e}_1)$, we obtain the required statement.

Now we are ready to prove

T h e o r e m 2. *If* $f\in L^1(\sigma)$, *then* $E_{2m}(C[f])\leq K(E_m^*(f)+\sum_{\nu=m+1}^{\infty}\nu^{-1}E_\nu^*(f))$ $(m=1,2,\ldots)$.

P r o o f. Let $\tau_m^*(f^\zeta)$ be the de la Vallée-Poussin mean of the Fourier series of f^ζ (see Lemma 4) and let

$$g_m^\zeta=\tau_{2m}^*(f^\zeta)-\tau_m^*(f^\zeta),\quad g_m(\zeta)=g_m^\zeta(0)$$

$$\mathrm{T}_m^\zeta=\tau_m^*(f^\zeta),\qquad \mathrm{T}_m(\zeta)=\mathrm{T}_m^\zeta(0)\quad(\zeta\in\boldsymbol{S}).$$

This g_m (T_m) is a measurable function on s: in fact this is clear, if f is continuous and for general f there is a sequence $\{h_k\}$ of continuous functions with $\|h_k-f\|_{L^1(\sigma)}\leq 2^{-k}$, which in turn implies that $\sum_{k=1}^{\infty}\|(h_k-f)^\zeta\|_{L^1_{2\pi}}<\infty$ for almost all $\zeta\in S$, and for such ζ's

$$\sum_{k=1}^{\infty}\|\tau_m^*(h_k^\zeta)-\tau_m^*(f^\zeta)\|_{L^1_{2\pi}}\leq 3\sum_{k=1}^{\infty}\|(h_k-f)^\zeta\|_{L^1_{2\pi}}<\infty,\text{ i. e.}$$

$$\tau_m^*(h_k^\zeta;\,x)\to\tau_m^*(f^\zeta;\,x)\text{ as }k\to\infty$$

a. e. for almost all $\zeta\in\boldsymbol{S}$, which already implies the measurability of g_m. Now since $\|\tau_m^*(f^\zeta)\|_{L^1_{2\pi}}\leq 3\|f^\zeta\|_{L^1_{2\pi}}$, the integrability of g_m also follows. Since g_m^ζ is a trigonometric polynomial of order at most $4m$ and since

$$\omega_1^\zeta(g_m;\,\delta)\leq\max_{\vartheta\in\boldsymbol{R}}|(g_m^\zeta)'(e^{i\vartheta})|\cdot\delta\leq 4m\cdot\delta\|(g_m^\zeta)\|_{C2\pi}$$

$$\leq 4m\,\delta(\|\tau_{2m}^*(f^\zeta)-f^\zeta\|_{C2\pi}+\|\tau_m^*(f^\zeta)-f^\zeta\|_{C2\pi})\leq 4\cdot 8m\delta E_m^*(f)\ (\zeta\in\boldsymbol{S}),$$

we obtain from Lemma 4 that

$$\omega_2^\zeta(C[g_m];\,\delta)\leq Km^2\delta^2E_m^*(f)\quad(\zeta\in\boldsymbol{S})$$

and so Corollary 1 of Assertion 5 gives $E_{m/2}(C[g_m]) \leq KE_m^*(f)$. Since the k-th Fourier coefficient of each g_m^ζ vanishes, when $k \leq m$ or $k > 4m$, we get that, if $C[g_m] = \Sigma_\alpha c_\alpha z^\alpha$, then $c_\alpha = 0$ provided $|\alpha| \leq m$ or $|\alpha| > 4m$ (see also the proof of Lemma 4).

Thus, $\tau_{m/2}(C[g_m]) = 0$ and consequently (see also the proof of Assertion 3)

$$\| C[g_m]) \| = \|C[g_m] - \tau_{m/2}(C[g_m]) \| \leq 4E_{m/2}(C[g_m]) \leq KE_m^*(f).$$

Let now m be arbitrary. The consideration at the beginning of the proof together with the fact $\| \tau_k^*(h) - h \|_{L^1_{2\pi}} = o(1)$ $(k \to \infty)$ give easily that $f = \mathrm{T}_m + \Sigma_{j=0}^\infty g_{2^j m}$ in $L^1(\sigma)$, by which $C[f] = C[\mathrm{T}_m] + \Sigma_{j=0}^\infty C[g_{2^j m}]$ and hence, since $C[\mathrm{T}_m]$ is a polynomial of order at most $2m$, $E_{2m}(C[f]) \leq \Sigma_{j=0}^\infty \| C[g_{2^j m}] \| \leq K \Sigma_{j=0}^\infty E_{2^j m}^*(f)$, which is exactly what we wanted to prove.

Let us list some corollaries.

Corollary 3. *If $f \in L^1(\sigma)$, then $\omega_r(C[f] : 1/m) \leq K_r(m^{-r} \Sigma_{\nu=0}^m \nu^{r-1} E_\nu^*(f) + \Sigma_{\nu=m+1}^\infty \nu^{-1} E_\nu^*(f))$ $(m = 1, 2 \ldots)$ for every $r \geq 1$.*

Put together Theorem 2 and Assertion 4.

Corollary 4. If *$f \in L^1(\sigma)$ and $r \geq 1$, then $\omega_r(C[f]; \delta) \leq K\delta^r \int_\delta^{2\pi} t^{-r-1} \omega_r^*(f; t)\, dt + \int_0^\delta t^{-1} \omega_r^*(f; t) dt$ especially*

$$\omega_1(C[f]; \delta) \leq K(\delta \int_\delta^{2\pi} t^{-2} \omega_1^*(f; t)\, dt + \int_0^\delta t^{-1} \omega_1^*(f; t) dt). \tag{8}$$

This follows from Corollary 3 and from Jackson's theorem ([4, p. 262]).

Corollary 5. *If $\Sigma_{\nu=1}^\infty \nu^{-1} E_\nu^*(f) < \infty$ and $f \in L^1(\sigma)$, then $C[f] \in A(\boldsymbol{B})$.*

Corollary 6. *If $E_\nu^*(f) \leq K\gamma^\nu$ $(0 < \gamma < 1)$ and $f \in L^1(\sigma)$, then $C[f]$ can be extended to a holomorphic function on the ball $\{z \mid |z| < 1/\sqrt{\gamma}\}$, especially it is analitic on $\bar{\boldsymbol{B}}$.*

Proof. By Theorem 2 $E_\nu(f) \leq K(\sqrt{\gamma})^\nu$ is satisfied, as well. Let P_ν be a polynomial of order at most ν with $\| P_\nu - f \| \leq K(\sqrt{\gamma})^\nu$. Putting $P_{-1} = 0$, we get

$$f = \sum_{\nu=-1}^\infty (P_{\nu+1} - P_\nu),$$

and since $\| P_{\nu+1} - P_\nu \| \leq K(\sqrt{\gamma})^\nu$, the series on the right converges uniformly in every ball $\{z \mid |z| \leq a/\sqrt{\gamma}\}$ $(0 < a < 1)$ (see [2, 1.5.2]).

Example. We show that Theorem 2 is the best possible, in a certain sense. We prove that (8) cannot be improved, namely: To every modulus of continuity ω there exists a function $f: \boldsymbol{S} \to \boldsymbol{C}$ with $\omega_1^*(f; \delta) \leq \omega(\delta)$ and with

$$\limsup_{\delta \to 0} \omega_1(C[f]; \delta)/(\delta \int_\delta^{2\pi} t^{-2} \omega(t)\, dt + \int_0^\delta t^{-1} \omega(t)\, dt) > 0.$$

Since to every modulus of continuity ω there is a concave modulus of continuity $\bar{\omega}$ with $\omega \leq \bar{\omega} \leq 2\omega$, we may assume ω to be concave. Let

$$g_1(x)=\sum_{\nu=1}^{\infty}(\omega(1/\nu)-\omega(1/(\nu+1)))\cos \nu x,$$

$$g_2(x)=\sum_{\nu=2}^{\infty}(\omega(1/\nu)-\nu^{-1}(\nu-1)\omega(1/(\nu-1))\sin \nu x.$$

By [7, p. 15.8] and [6, p. 67] $\omega(g_j;\ \delta)\leq K\omega(\delta)\ (j=1,\ 2)$. Let $P[g_j](re^{it})$ be the Poisson-integral of g_j, e. g. $P[g_1](re^{it})=\Sigma_{\nu=0}^{\infty} r^\nu(\omega(1/\nu)-\omega(1/(\nu+1))\cos \nu t$. Finally, lef $f_j;\ \boldsymbol{S}\to\boldsymbol{C}$ be defined by $f_j(\zeta)=P[g_j](\zeta_n)$ and let

$$h_1(z)=P[g_1](z)+iP[\tilde{g}_1](z)=\sum_{\nu=1}^{\infty}(\omega(1/\nu)-\omega((\nu+1)^{-1})z^\nu \quad (z\in \boldsymbol{U}),$$

$$h_2(z)=P[g_2](z)+iP[\tilde{g}_2](z)=-i\sum_{\nu=2}^{\infty}(\omega(1/\nu)-\nu^{-1}(\nu-1)\,\omega(1/(\nu-1))z^\nu \quad (z\in \boldsymbol{U}),$$

$$H_j(z)=h_j(z_n) \quad (z\in \overline{\boldsymbol{B}},\quad j=1,\ 2).$$

Since the Poisson-kernel is non-negative, $\omega(P[g_j](re^{i}\cdot);\ \delta)\leq K\omega(\delta)$ is also satisfied, and thus $\omega_1^*(f_j;\ \delta)\leq K\omega(\delta)$. On the other hand, the expression of $C[f_j]$ shows that $C[f_j]=C[H_j]=H_j$, and so $C[f_j](\lambda e_n)=h_j(\lambda)\ (\lambda\in\boldsymbol{U}, i=1, 2)$. A simple computation shows that

$$\omega(\tilde{g}_1;\ \delta)\geq c\,\delta\int_\delta^{2\pi} t^{-2}\omega(t)\,dt-O(\omega(\delta)),$$

$$\omega(\tilde{g}_2;\ \delta)\geq c\int_0^\delta t^{-1}\omega(t)\,dt-O(\omega(\delta)),$$

and thus the same estimate holds for $\omega_1(h_j;\ \delta)$, which proves our statement in the case $\delta\int_\delta^{2\pi}t^{-2}\,\omega(t)\,dt\neq O(\omega(\delta))$, or $\int_0^\delta t^{-1}\ \omega(t)\,dt\neq O(\omega(\delta))$. If, however, $\delta\int_\delta^{2\pi} t^{-2}\,\omega(t)\,dt=O(\omega(\delta))$, then the function $g_2^*(x)=\Sigma_{\nu=1}^{\infty}\ \nu^{-1}\omega(\nu^{-1})\sin \nu x$ satisfies $\omega(g_2^*;\ \delta)\leq K\omega(\delta)$ (compare with the proof of [7, Lemma 4]) and since $\omega(g_2^*;\ \delta)\geq c\int_0^\delta t^{-1}\ \omega(t)\,dt$ (see also [5, pp. 241, 242]), the above procedure with g_2^* in the place of g_2 yields a suitable function f_2^*.

Clearly the method of the above example can be applied in many other cases, e. g. to prove that in Corollary 5 the assumption $\Sigma_{\nu=1}^{\infty}\nu^{-1}E_\nu^*(f)<\infty$ is necessary.

Remark. Most of our results can be extended to the L^p-case. In fact the statements and the proofs are equally true, if we exchange $\|\cdot\|$, $E_m(f)$, $E_m^*(f)$ etc. by $\|f\|_{L^p(\sigma)}=\{\int_{\boldsymbol{S}}|f(z)|^p d\sigma(z)\}^{1/p}$, $E_m(f)_{L^p}=\inf_{P_n}\|f-P_n\|_{L^p(\sigma)}$, $E_m^*(f)_{L^p}=\sup_{\zeta\in \boldsymbol{S}}E_m^T(f^\zeta)_{L^p_{2\pi}}$.

E. g. the analogue of Theorem 2 reads as

Theorem 2*. *If* $f\in L^p(\sigma)$, *then* $E_{2m}(C[f])_{L^p}\leq K(E_m^*(f)_{L^p}+\Sigma_{\nu=m+1}^{\infty}\nu^{-1}E_\nu^*(f)_{L^p})$ $(m=1,\ 2,\dots)$.

Since the formulation and the proof of these statements are almost word by word, we do not go into details.

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Bolyai Institute
Szeged Hungary

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