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DIVERGENCE OF LAGRANGE INTERPOLATION (COMPLEX AND TRIGONOMETRIC CASES)*

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Summary. In a previous joint paper with P. Erdös we proved the following result (see [1]). If $X=\{x_{kn}\}$, $n=1, 2, \ldots$, $1\leqq k\leqq n$, $-1\leqq x_{nn}<x_{n-1,n}<\ldots<x_{1n}\leqq 1$ is a triangular matrix of interpolation, then there is a continuous function $F(x)$ on $[-1, 1]$, so that for the sequence of Lagrange interpolatory polynomials $L_n(F, X, x)$ of degree $n-1$ $\overline{\lim}_{n\to\infty}|L_n(F, X, x)|=\infty$ for almost all x in $[-1, 1]$. Now I am going to settle the corresponding problem for complex domain. The notations and the proof are analogous to the paper cited above, but, of course, some other difficulties had to be overcome.

At the same time I shall obtain results for the trigonometric case, too.

1. Preliminary Results. A detailed list of the corresponding results for the real case can be found in [1]. Here we restrict ourselves only to the complex case. First of all let us see the definitions and notations. Let $Z=\{z_{kn}=\exp(i\Theta_{kn})\}$, $k=1, 2, \ldots, n$, $n=1, 2, \ldots$, with

$$0\leqq\Theta_{1n}<\Theta_{2n}<\cdots<\Theta_{nn}<2\pi, \quad n=1, 2, \ldots, \tag{1.1}$$

be a triangular matrix on the unit-circle line $\Gamma=\{z;\ z=\exp(i\Theta), 0\leqq\Theta<2\pi\}$. Let, sometimes omitting the superfluous notations,

$$L_n(f, Z, z)=L_n(f, z)=\sum_{k=1}^{n} f(z_{kn})\, l_{kn}(z)$$

for $f(z)\in AC$ (i. e. $f(z)$ is analytic on $U=\{z;\ |z|<1\}$ and continuous on the closure $[U]=\{z;\ |z|\leqq 1\}$) be the corresponding Lagrange interpolatory polynomials of degree $\leqq n-1$. Here, as usual,

$$l_{kn}(z)=l_{kn}(Z, z)=\prod_{j\neq k}(z-z_{jn})/(z_{kn}-z_{jn}) \quad (k=1, 2, \ldots n)$$

are the corresponding fundamental polynomials. We often use the so-called Lebesgue functions and constants, which are

$$\lambda_n(z)=\lambda_n(Z, z)=\sum_{k=1}^{n}|l_{kn}(z)|; \quad \lambda_n(Z)=\lambda_n=\max_{|z|\leqq 1}\lambda_n(z).$$

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* The results were announced in Oberwolfach, August 9—16, 1980. The detailed proof appeared in *Acta Math. Acad. Sci. Hungar*, **39**, 1982, 367—377.

In his paper Alper [2] proved that for any matrix Z $\lambda_n > (\ln n)/8\pi^{1/2}$, from where he obtained that one can find a function $f_1 \in AC$ such that $\overline{\lim}_{n\to\infty} \| L_n(f_1, Z, z) \| = \infty$, where $\| g(z) \| = \max \{ | g(z) | : |z| = 1 \}$.

In a very recent paper German [3] — among others — proved that for certain matrices Z, $\overline{\lim}_{n\to\infty} | L_n(f, Z, z) | = \infty$ a. e. on Γ for a suitable $f \in AC$.

2. Results. 2.1. First we state the following result.

Theorem 2.1. *For any matrix Z one can find a function $f(z) \in AC$ such that*

$$\overline{\lim_{n\to\infty}} | L_n(F, Z, z) | = \infty \quad \text{a. e. on } \Gamma. \tag{2.1}$$

2.2. Now we settle the trigonometric case, too. Let n be odd number and let us denote for a 2π periodic continuous $f(\Theta)$ (shortly $f \in \tilde{C}$) the corresponding trigonometric interpolatory polynomial of degree $\leqq n$, based on the nodes (1.1) by $L_n(f, \Theta, \theta)$. Then

Theorem 2.2. *For any matrix $\{\Theta_{kn}\}$, $k = 1, 2, \ldots, n$, $n = 1, 3, 5, \ldots$, one can find a function $f \in \tilde{C}$, for which $\overline{\lim}_{n\to\infty} | L_n(f, \Theta, \theta) | = \infty$ a.e. on the real line.*

3. On the Proofs. 3.1. As we mentioned, the ideas are very similar to the ones applied for the real case. In what follows let $\Theta_{0n} = \Theta_{n+1,n} = 0$ and $\Delta_{kn} = \Delta\Theta_{kn} = \Theta_{k+1,n} - \Theta_{kn}$ $(k = 0, 1, \ldots, n;\ n = 1, 2, \ldots)$. For the sake of simplicity we suppose that

$$\Delta_{kn} \leqq \delta_n \overset{\text{def}}{=} 1/\ln n. \tag{3.1}$$

The following result plays a fundamental role in the proof: If we define the intervals I_{jm} by

$$I_{jm} = [(2\pi/m)(j-1),\ 2\pi j/m), \quad j = 1, 2, \ldots, m,$$

then for the points of Γ we state the next relation, which is interesting in itself (see [1, Lemma 4.1]).

Lemma 3.1. *Let $A > 0$ be arbitrary fixed number. Then, considering the arbitrary integer $m \geqq \max[\exp(8A^3),\ \exp(\exp 100)] \overset{\text{def}}{=} m_0(A)$, for any $n \geqq n_0(m)$ there exists the set $H_{1n} \subset \Gamma$ with $\mu(H_{1n}) \leqq 1/\ln\ln m$, further whenever $\Theta \in \Gamma \setminus H_{1n}$,*

$$\sum_{k \in Q} | l_{kn}(e^{i\Theta}) | \geqq (\ln m)^{1/3} \geqq 2A \quad \text{if} \quad n \geqq n_0(m).$$

Here Q means those indices k, $1 \leqq k \leqq n$, for which $\Theta_{kn} \notin I_{j(\Theta),m}$, $k \notin K_{3n}$, where K_{3n} is a certain index-set, having $\sqrt{\ln m}$ elements at most; $I_{j(\Theta),m}$ means the interval, containing Θ. $\mu(\cdots)$ stands for the Lebesgue measure.

3.2. Using this lemma and further considerations, we obtain with a well defined p the functions $f^{[1]}, f^{[2]}, \ldots, f^{[p]} \in AC$ and the sets $G^{[1]}, G^{[2]}, \ldots, G^{[p]} \subset [0, 2\pi)$ (see the corresponding parts in [1]) such that

$$\overline{\lim}_{n\to\infty} | L_n(f^{[j]}, e^{i\Theta}) | = \infty \quad \text{if} \quad \Theta \in G^{[j]},$$

where $1 \leqq j \leqq p$ (see [1, 4.34)]).

To define the proper (linear) combination of the functions $f^{[j]}$ on $\Theta \cup G^{[j]}$ we state

Lemma 3.2. *If* $r_1(z) \in AC$ *and* $r_2(z) \in AC$, *moreover,*

$$\overline{\lim_{n\to\infty}} |L_n(r_1, e^{i\Theta})| = \infty \quad if \quad \Theta \in B_1, \quad B_1 \subset [0, 2\pi),$$

$$\overline{\lim_{n\to\infty}} |L_n(r_2, e^{i\Theta})| = \infty \quad if \quad \Theta \in B_2, \quad B_2 \subset [0, 2\pi),$$

then any fixed interval (β_1, β_2) $(\beta_1 < \beta_2)$ *contains an* α *such that*

$$\overline{\lim}_{n\to\infty} |L_n(\alpha r_1 + r_2, e^{i\Theta})| = \infty \text{ a. e. on } B_1 \cup B_2.$$

For the proof see [1, 4.4.7].

By Lemma 3.2 we get: For arbitrary fixed $\rho > 0$ there exists the function $F_\rho(z) \in AC$, $\|F_\rho\| \leq 16$ such that $\overline{\lim}_{n\to\infty} |L_n(F_\rho, e^{i\Theta})| = \infty$ a. e. on $P_\rho \subset [0, 2\pi)$, where $\mu(P_\rho) \geq 2\pi - \rho$.

3.3. By this it is not so difficult to obtain (2.1).

3.4. Using more complicated argument, we can prove the theorem omitting the restriction (3.1).

3.5. After having settled the complex case it is easy to verify Theorem 1.2

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