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ON APPROXIMATION BY COMPLEX SPLINES

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Summary. The paper is concerned with the space $S^r_{\Delta_N}(\Gamma)$ splines in the complex variable z of order r w. r. t. a given partition Δ_N of a rectifiable Jordan curve Γ.

Let $s_{f,N}$ be the L_2-approximation to $f \in C(\Gamma)$ onto $S^r_{\Delta_N}(\Gamma)$ w. r. t. the following scalar product $(f, g)=\int_\Gamma f\bar{g}\,|dt|$, and let $S_{f,N}$ be the associated with $s_{f,N}$ analytic spline, defined interior Γ by means of the Cauchy integral.

The aim of the paper is to give estimates of the convergence of the functions $s_{f,N}$ $S_{f,N}$ and their derivatives in L_∞-norm for the situation, in which the approximated function is analytic interior Γ and of class C^k $(k=0, \ldots, r-1)$ on Γ.

The results obtained are a generalization of some properties of splines on the real axis.

1. Introduction. The purpose of this paper is to give some properties of complex splines on a rectifiable Jordan curve Γ and the application of it to the approximation of a function analytic interior Γ and smooth in the corresponding closed region. In the paper we shall give a generalization from some properties of splines given by de Boor and Ciesielski [2, 3, 4] and earlier for dyadic partitions by Ciesielski and Domsta [5] to splines in the complex variable z.

Let Γ be a rectifiable Jordan curve and let $t_1, t_2, \ldots, t_N$ be points on Γ, arranged in counterclockwise order, separating Γ into arcs Γ_j. $j=1, \ldots, N$ with $\Gamma_j=\widehat{t_{j-1}, t_j}$ the arc from t_{j-1} to t_j, $t_j=N_{N+j}$. Denote this partition of Γ by Δ. The function $s_\Delta \in C^{r-2}(\Gamma)$ is called a complex spline of order r with respect to the partition Δ, where $r>1$, if it is on each arc Γ_j a polynomial of degree at most $r-1$ in the complex variable z. Let $S^r_\Delta(\Gamma)$ be the set of all complex splines of order r w. r. t. the partition Δ.

The function S_Δ, defined in the interior of Γ by the Cauchy integral

$$S_\Delta(z)=(1/2\pi i)\int_\Gamma (t-z)^{-1}\, s_\Delta(t)dt,$$

is said to be an analytic spline associated with the function s_Δ. Because the function s_Δ satisfies Lipschitz's condition on Γ, so we can define the function S_Δ on Γ as the limiting value [8, 10] for approach from within

$$S_\Delta(t)=(1/2)\, s_\Delta(t)+(1/2\pi i)\int_\Gamma (x-t)^{-1} s_\Delta(x)dx, \quad t \in \Gamma.$$

Analytic splines, defined in this way, were introduced by Ahlberg, Nilson and Walsh in [1]. Another definition of analytic splines was given in [13].

The modulus of continuity of the function f, defined on Γ, is defined by the formula $\omega(f, h) = \sup\{|f(x)-f(y)| : 0 \leq |x-y| \leq h, x, y \in \Gamma\}$. The derivative of the function f, defined on Γ, we define as usual $f'(z) = \lim_{t\to z}[f(t) - f(z)]/(t-z)$, $t \in \Gamma$.

A Jordan curve Γ is called a curve of class $S_{h,\lambda}$, if for every two points $x, y \in \Gamma$, separating Γ into arcs Γ_1 and Γ_2, such that $|x-y| \leq h$, $\min(|\Gamma_1|, |\Gamma_2|) \leq \lambda |x-y|$, where $|\Gamma_j|$ is the length of Γ_j. Further we shall investigate complex splines, defined on curves of class $S_{h,\lambda}$ with $r\|\Delta\| \leq h$, where $\|\Delta\| = \max\{|t_j - t_{j-1}| : j = 1, \ldots, N\}$. $\|f\|_p = (\int_\Gamma |f|^p dt)^{1/p}$ and $\|f\| = \|f\|_\infty = \sup_{z\in\Gamma}|f(z)|$ are the norms in the spaces $L_p(\Gamma)$, $1 \leq p < \infty$ and $C(\Gamma)$ respectively.

2. Divided Differences and B-splines. For every function $f: \Delta \to \mathbf{C}$ the divided differences at the points $t_0, \ldots, t_n$, $0 \leq n < N$ are defined as follows (see [6, 4, 12]): $[t_0; f] = f(t_0)$, $[t_0, \ldots, t_n; f] = ([t_0, \ldots, t_{n-1}; f] - [t_1, \ldots, t_n; f])/(t_0 - t_n)$, $n = 1, 2, \ldots$. Let $f \in C^r(\Gamma)$. For this function we write the Taylor formula at the point t_j with integral remainder

$$f(t) = \sum_{k=0}^{r-1} D^k f(t_j)(t-t_j)^k/(k!) + (1/(r-1)!) \int_{t_j}^{t} D^r f(u)(t-u)^{r-1} du$$

$$= \sum_{k=0}^{r-1} D^k f(t_j)(t-t_j)^k/(k!) + (1/r!) \int_{t_j}^{t_{j+r}} D^r f(u)(t-u)_+^{r-1} du,$$

where $(t-u)_+^{r-1} = (t-u)^{r-1}$, if two points u and t are oriented as the curve Γ and 0 otherwise, and D is the differentiation operator. From this we obtain

$$(1) \qquad [t_j, \ldots, t_{j+r}; f] = (1/r!) \int_{t_j}^{t_{j+r}} D^r f(u) M_{j,r}(u) du, \text{ where}$$

$$M_{j,r}(u) = [t_j, \ldots, t_{j+r}; r(t-u)_+^{r-1}].$$

The function $M_{j,r}$ is called the j-th B-spline of order r w. r. t. Δ. The j-th normalized B-spline is defined as follows:

$$(2) \qquad N_{j,r}(u) = (1/r)(t_{j+r} - t_j) M_{j,r}(u).$$

Analogously as for splines on the real axis we can prove the following properties of B-splines and theorems:

$$(\text{M.1}) \qquad M_{j,r}(t) = [r/(r-1)]\{[(t-t_j)/(t_{j+r}-t_j)]M_{j,r-1}(t) + [(t_{j+r}-t)/(t_{j+r}-t_j)]M_{j+1,r-1}(t)\},$$

$$(\text{N.1}) \qquad N_{j,r}(t) = [(t-t_j)/(t_{j+r-1}-t_j)]N_{j,r-1}(t) + [(t_{j+r}-t)/(t_{j+r}-t_{j+1})]N_{j+1,r-1}(t),$$

(M.2) $\operatorname{supp} M_{j,r} = \widehat{t_j, t_{j+r}}$,

(N.2) $\operatorname{supp} N_{j,r} = \widehat{t_j, t_{j+r}}$,

(M.3) $\| M_{j,r} \| \leq r 2^{r-2} \lambda^{r-1} | t_{j+r} - t_j |^{-1}$,

(N.3) $\| N_{j,r} \| \leq 2^{r-2} \lambda^{r-1}$,

(M.4) $\int_\Gamma M_{j,r}(t) dt = 1$,

(N.4) $\sum_{j=1}^{N} N_{j,r}(t) = 1$.

Theorem 1. *For $r>1$ and for arbitrary a_j, $j=1, \ldots, N$, we have*

$$D \sum_{j=1}^{N} a_j N_{j,r} = \sum_{j=1}^{N} (a_j - a_{j-1}) M_{j,r} \quad (a_0 = a_N).$$

Theorem 2. *If for $r>1$ $\sum_{j=1}^{N} a_j N_{j,r} = 0$ on the arc $\gamma \subset \Gamma$, then $a_j = 0$ for $j \in E$, where $E = \{ j : (\operatorname{supp} N_{j,r}) \cap \gamma$ contains more than one point.*

Corollary. *The system of functions $\{N_{j,r}\}$, $j=1, \ldots, N$, $r>1$, is a basis in the space $S^r_\Delta(\Gamma)$.*

3. Orthogonal Projections onto the Space $S^r_\Delta(\Gamma)$. Let $N_{i,r,p} = N_{i,r} | r / (t_{i+r} - t_i) |^{1/p}$. Analogously as for splines on the real axis we can prove the following inequalities:

$$(r\lambda)^{-1/q} \leq \| N_{i,r,p} \|_p \leq r 2^{r-2} \lambda^r, \quad p^{-1} + q^{-1} = 1,$$

$$\| \sum_{i=1}^{N} a_i N_{i,r,p} \|_p \leq p 2^{r-2} \lambda^{r-1/q} \| a \|_p,$$

where $\| a \|_p = (\sum_{i=1}^{N} |a_i|^p)^{1/p}$.
Put

$$\psi_{r,i}(t) = (t - t_{i+1}) \cdot \ldots \cdot (t - t_{i+r-1}) / (r-1)!,$$

$$\psi^+_{r,i}(t) = (t - t_{i+1})_+ \cdot \ldots \cdot (t - t_{i+r-1})_+ / (r-1)!.$$

In the space $L_2(\Gamma)$ the following scalar product is given: $(f, g) = \int_\Gamma f(t) \overline{g(t)} \, | dt |$. Let $H^r = H^r(\Gamma)$ denote the space of functions u, defined on Γ, such that u has absolutely continuous derivative of order $r-1$ and $D^r u \in L_2(\Gamma)$.

Analogously as in [3] and [4] we can prove the following lemmas:

Lemma 1. *For $r>1$ and $i, j = 1, \ldots, N$ we have* $[t, \ldots, t_{j+1}; \psi^+_{r,i}] = [(r-1)! \, (t_{j+1} - t_j)]^{-1} \delta_{ij}$.

Lemma 2. *Let $u \in L_2(\Gamma)$ and $\operatorname{supp} u \subset \widehat{t_j, t_{j+r}}$. Then for a given integer j $\int_\Gamma N_{i,r}(t) u(t) dt = \delta_{ij}$, $i = 1, \ldots, N$, if and only if $u = D^r f$ for some $f \in H^r(\widehat{t_{j-r+1}, t_{j+2r-1}})$ $f(t_k) = \psi^+_{r,j}(t_k)$* for $k = j-r+1, \ldots, j+2r-1$.

Further we need the following

Theorem 3 (see [9, 11]). *Let E be a closed limited point set, containing more than one point such that $\mathbf{C}\setminus E$ is connected. Then for any polynomial P_n of degree n, $\|P'_n\|_E \leq 2en^2d^{-1}\|P_n\|_E$, where d is the diameter of E and $\|f\|_E = \sup_{z\in E}|f(z)|$.*

Lemma 3. *For any integer j, $a\in\Gamma_j$ and $b\in\Gamma_{j+r-1}$, there exist: a constant $d=d(r,\lambda)$, depending only on r and λ, and a function $h\in L_2(\Gamma)$ such that* $\operatorname{supp} h\subset\widehat{a,b}$, $\|h\|\leq d|b-a|^{-1}$, $\int_\Gamma h(t)N_{i,r}(t)dt=\delta_{ij}$, $i=1,\dots,N$.

Proof. Divide the arc $\widehat{a,b}$ by the points $a=t'_0, t'_1,\dots,t'_r=b$ such that $|\widehat{t'_{i-1},t'_i}|=|\widehat{a,b}|r^{-1}$, $i=1,\dots,r$. Let M be the B-spline of order r w. r. t. the partition $\Delta'=\{t'_0,\dots,t'_r\}$ and let $w(t)=\int_a^t M(x)dx$. Now we define the function $f(t)=w(t)\psi_{r,j}(t)$ for $t\in\widehat{a,b}$, $f(t)=0$ for $t\in\widehat{t_{j-r+1},a}$ and $f(t)=\psi_{r,j}(t)$ for $t\in\widehat{b,t_{j+2r-1}}$. The function $h=D^rf$ satisfies the assumption of Lemma 2. It remains to verify an estimate of the norm of the function h. From (M.3), definition of w and Theorem 3 we deduce that there exist constants c and d, depending only on r and λ such that $\|D^jw\|\leq c|b-a|^{-j}$, $\|D^{r-j}\psi_{r,i}\|\leq d|b-a|^{j-1}$ and $D^r\psi_{r,j}=0$. Hence we obtain

$$\|h\|\leq\sum_{j=1}^{r}\binom{r}{j}\|D^jw\|\,\|D^{r-j}\psi_{r,j}\|\leq d|b-a|^{-1}.$$

Applying Lemmas 1, 2 and 3 we can prove the following theorem, analogous to the real case (see [2, 3, 4]).

Theorem 4. *There exist constants c and d, depending only on r and λ such that for $a=\{a_i\}_{i=1}^N\in l_p$, $1\leq p\leq\infty$,*

$$c\|a\|_p\leq\Big\|\sum_{i=1}^N a_iN_{i,r,p}\Big\|_p\leq d\|a\|_p.$$

For given N, p and $N\geq1$, $1\leq p\leq\infty$, denote by l_p^N the space $\mathbf{C}^N$ with the norm $\|a\|_p=(\Sigma_{i=1}^N|a_i|^p)^{1/p}$, $\|a\|=\max(|a_i|:1\leq i\leq N)$. The norm of a matrix $A=[a_{ij}]$, $i,j=1,\dots,N$ is defined as follows

$$\|A\|_p=\sup\{\|Ax\|_p : x\in l_p^N,\ \|x\|_p=1\}.$$

Further we need the following theorem, given in [6] (see also [4]).

Theorem 5. *Let $A=[a_{ij}]$, $i,j=1,\dots,N$ be a band matrix with $a_{ij}=0$ for $d(i,j)=\min(N-|i-j|,|i-j|)\geq r$. Assume that $\|A\|_p\leq1$ and $\|A^{-1}\|_p\leq M$ for some $1\leq p\leq\infty$ and some $M>0$. Then with $A^{-1}=[b_{ij}]$ there exist constants $C=C(r,M)$ and $q=q(r,M)$, $0<q<1$ depending only on r and M such that $|b_{ij}|\leq Cq^{d(i,j)}$, $i,j=1,\dots,N$.*

Let P_Δ^r be the orthogonal projection of the space $L_2(\Gamma)$ onto the subspace $S_\Delta^r(\Gamma)$. Hence from the Corollary of Theorem 2 we deduce that for any function $f\in L_2(\Gamma)$ the spline $s=P_\Delta^rf$ is determined uniquely by the system of equations $(f,N_{i,r,2})=(s,N_{i,r,2})$, $i=1,\dots,N$. Putting $s=\Sigma_{j=1}^N a_jN_{j,r,2}$, we have

$$\sum_{j=1}^N a_j(N_{j,r,2},N_{i,r,2})=(f,N_{i,r,2}),\qquad i=1,\dots,N.$$

$A=[(N_{i,r,2}, N_{j,r,2})]$ $i, j=1, \ldots, N$ is the Gramian matrix and then it is non-singular. Applying Theorem 5, analogously as in the real case (see [3, 4]) we can prove that for the matrix $B=A^{-1}=[b_{ij}]$ we have $|b_{ij}|\leq Cq^{d(i,j)}$, where $C=C(r, \lambda)>0$ and $0<q=q(r, \lambda)<1$. Hence we obtain

Theorem 6. *There exists a constant C, depending only on r and λ, such that for the operator P^r_Δ as a map on $C(\Gamma)$ we have*

$$\|P^r_\Delta\|\leq CR, \tag{3}$$

where $R^2=\max_{i,j}|t_{i+r}-t_i|/|t_{j+r}-t_j|$.

4. Best Approximation by Splines. Further we need the following lemmas:

Lemma 1. *Let $f\in C(\Gamma)$. Then there exist constants $c=c(k, \lambda)$ and $d=d(k, \lambda)$ and a spline g_k of order k w. t. r. Δ satisfying the following conditions:*

$$\|g_k\|\leq c\|f\|, \tag{4}$$

$$\|f-g_k\|\leq d\omega(f, \|\Delta\|).$$

Proof. Let $\Delta'=\{t'_1, \ldots, t'_{mk}\}$ $(t'_{mk}=t_0=t_N)$ be a partition of Γ such that each point $t'_i\subset\Delta$ and $\|\Delta'\|\leq 2\|\Delta\|$. Put $\varphi_j(t)=\int_{t'_{jk}}^{t} M_{jk,k}(u)du$, where $M_{jk,k}$ is the B-spline, defined by (1) for the partition Δ'. It follows from (M.2), (M.3), (M.4) and the definition of the modulus of continuity that the function $g_k(t)=f(x_j)+[f(x_{j+1})-f(x_j)]\,\varphi_j(t)$ for $t\in \widehat{x_j, x_{j+1}}, j=0, \ldots, m$, where $x_j=t_{jk}$, satisfies (4).

Let $m=\min\{|t_{i+1}-t_i|: i=1, \ldots, N\}$ and $K=|\Delta\|/m$.

Lemma 2. *If s is a spline of order m w. r. t. Δ such that for $f'=g\in C^j(\Gamma)$, $0\leq j<m$, $\|g^{(i)}-s^{(i)}\|\leq C\|\Delta\|^{j-i}\omega(g^{(j)}, \|\Delta\|)$, $1\leq i\leq j$, $C=\text{const}$, then there exists a spline S of order $m+1$ w. r. t. Δ and a constant B, depending only on m, K and λ such that for $0\leq i\leq j$, $1\leq j\leq m$*

$$\|f^{(i)}-S^{(i)}\|\leq B\|\Delta\|^{j-i}\omega(f^{(j)}, \|\Delta\|). \tag{5}$$

Proof. We shall construct a function S analogously as in [4]. Let $\Delta'=\{t'_1, \ldots, t'_{nm}\}$, $(t'_{nm}=t_0=t_N)$ be a partition of Γ such that each point $t'_i\in\Delta$ and $\|\Delta'\|\leq 2\|\Delta\|$. Put $g(t)=s(t)+\Sigma_{k=0}^{n-1} a_k N_k(t)$, where $N_k=N_{km,m}$ is the normalized B-spline defined for the partition Δ' by (2), $a_k=r(x_{k+1}-x_k)^{-1}\int_{x_k}^{x_{k+1}}[f'(x)-s(x)]dx$, $x_k=t'_{km}$. It follows from (N.2), (N.3), (M.4) and Theorem 3 that the function g satisfies the following conditions: $\|f^{(i)}-g^{(i)}\|\leq C_1\|\Delta\|^{j-i}\omega(f^{(j)}, \|\Delta\|)$, $1\leq i\leq j$, where C_1 depends only on m, K and λ, and $\int_{x_k}^{x_{k+1}} g(x)\,dx=\int_{x_k}^{x_{k+1}} f'(x)dx$, $k=0, \ldots, n-1$. Hence the function $S(t)=f(t_0)+\int_{t_0}^{t} g(x)dx$ satisfies (5).

Theorem 7. *Let $f\in C^j(\Gamma)$, $0\leq j\leq r-1$ and $s_f=P^r_\Delta f$. Then*

$$\|f^{(i)}-s_f^{(i)}\|\leq C\|\Delta\|^{j-i}\omega(f^{(j)}, \|\Delta\|),\quad 0\leq i\leq j, \tag{6}$$

where C is a constant depending only on r, K and λ.

P r o o f. Let $g=g_r$ be a spline from Lemma 1. Applying (3) we obtain

$$\|f-s_f\|\leq\|f-g\|+\|g-s_f\|=\|f-g\|+\|s_{f-g}\|\leq C\omega(f, \|\Delta\|).$$

Let s be a spline of degree 1 w. r. t. Δ interpolating f at Δ. Then $\|f^{(i)}-s^{(i)}\|\leq C_1\omega(f^{(i)}, \|\Delta\|)$, $i=0, 1$. Hence by Theorem 3, $\|s'_f-s'\|\leq C_1 m^{-1}\|s_f-s\|\leq C_2 m^{-1}\omega(f, \|\Delta\|)\leq C_3 K\|f'\|$ and we have

$$\|s'_f\|\leq C\|f'\|. \tag{7}$$

Let now $g=g_{r-1}$ be a spline from Lemma 1. It follows from Lemma 2 that there exists a spline S of order r w. r. t. Δ such that

$$\|f^{(i)}-S^{(i)}\|\leq B\|\Delta\|^{1-i}\omega(f', \|\Delta\|), \quad i=0, 1.$$

Hence by (3) and (7) we have for $i=0, 1$

$$\|f^{(i)}-s_f^{(i)}\|\leq\|f^{(i)}-S^{(i)}\|+\|S^{(i)}-s_f^{(i)}\|=\|f^{(i)}-S^{(i)}\|$$

$$+\|s_{f-S}^{(i)}\|\leq C\|\Delta\|^{1-i}\omega(f', \|\Delta\|).$$

We obtain the remaining inequalities of (6) analogously.

5. Approximation by Analytic Splines. Let f be an analytic function in the region $D=\operatorname{int}\Gamma$ and of class C^j in $\bar{D}$, $0\leq j\leq r-1$ and let $s_f=P^r_\Delta f$. Put

$$S_f(z)=(1/2\pi i)\int_\Gamma (t-z)^{-1}s_f(t)dt, \quad z\in D.$$

The function S_f is analytic in D and of class C^{r-2} in $\bar{D}$. Applying Theorem 7, we can prove the following theorem analogously as in [14].

T h e o r e m 8. *There exists a constant C, depending only on r, K and λ such that*

$$\|f-S_f\|\leq C\|\Delta\|^j|\ln\|\Delta\||\,\omega(f^{(j)}, \|\Delta\|), \quad 0\leq j\leq r-1,$$

$$\|f^{(i)}-S_f^{(i)}\|\leq C\|\Delta\|^{j-i}\omega(f^{(j)}, \|\Delta\|), \quad 1\leq i\leq j\leq r-1, \ i\leq r-2.$$

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