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A MODULAR SPACE OVER A FIELD WITH VALUATION

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Summary. In the present paper we introduce a modular ρ, defined on a vector space X over a field K with a valuation $|\cdot|$. This modular generates a F-quasi-norm $|.|_\rho$ in certain subspaces X_ρ of the space X with constant $k=\inf\{|\alpha|>1:\alpha\in K\}$ in the triangle inequality. Moreover, we prove that the F-quasi-norm satisfies the condition (B), which implies the condition (F5). In the case of convex modular the functional $\|\cdot\|_\rho$ is a quasi-norm with the same constant k. We give examples illustrating the fact that, in general, the triangle inequality with constant $k_0=1$ is not satisfied. Next, we give some examples of $\widetilde{F}$-norms and F-norms not satisfying the condition (B). In the case of K real or complex numbers we obtain the modulars investigated in the papers [4] and [3].

1. Let X be a vector space over a field K with the identity element e and valuation $|\cdot|:K\to \boldsymbol{R}$ (where $\boldsymbol{R}$ denotes the real numbers). The set of real numbers, $|\alpha|$, $\alpha\in K$, will be called the *set of values* of K or just the *values* of K and will be denoted by $|K|$. We already noted that the values $|K^*|$, where K^* denotes the nonzero elements of K, form a multiplicative subgroup of the positive reals $(\boldsymbol{R}^+,\cdot)$ However, as is well-known, $(\boldsymbol{R}^+,\cdot)$ has only two types of subgroups — they are either cyclic or dense in $\boldsymbol{R}^+$ (see [1]). If $|K^*|$ is an infinite cyclic group, the valuation is called *discrete*; equivalently, K is said to be *discretely valued*. It is clear that, if the valuation is discrete, then 0 is the only limit point of $|K^*|$. Next theorem (Theorem 5, p. 8 [5]) shows that the converse of this statement is also true: "If 0 is the only limit point of $|K|$, then K must be discretely valued, moreover, in this case, the generator of $|K^*|$ is $r\in|K|$ such that $r>1$ and the distance from r to 1 is minimal", i. e. $r=\inf\{|\alpha|>1:\alpha\in K\}$. From this comment it follows ([5]) that the valuation is either discrete or "dense" in $\boldsymbol{R}^+$.

Let Q be the set of rational numbers. For $a/b\in Q$ and any fixed prime p, write $a/b=p^s(c/d)$ where neither c nor d is devisible by p. The (normalized) *p-adic valuation* of $|a/b|$ is defined to be p^{-s}. Let Q_p (the p-adic numbers) denote the complection of Q with respect to p-adic valuation ([5]). Let K be a field with a valuation $|\cdot|$. If instead of the strong triangle inequality is satisfied $|\alpha+\beta|\leq\max(|\alpha|,|\beta|)$, the valuation is said to be *nonarchimedean*, otherwise it is called *archimedean*. We recall that a discrete valuation must be nonarchimedean and, if K is archimedean valued, $|K|$ must be dense in $\boldsymbol{R}^+$. In this paper $\boldsymbol{N}$ denote the natural numbers and $\boldsymbol{Z}$ denote the whole numbers.

2. Given a vector space X over a field K with nontrivial valuation $|\cdot|$. We say that the mapping $\|\cdot\|: X \to \boldsymbol{R}$ is a $\tilde{F}$-quasi-norm, if it has the following properties:

(F1) $\|x\|=0$ if and only if $x=0$;

(F2) $\|\lambda x\| \leqq \|x\|$ for all $\lambda \in K$, $|\lambda| \leqq 1$,

(F3) There is a $k \geqq 1$, for which $\|x+y\| \leqq k(\|x\|+\|y\|)$ for all $x, y \in X$,

(F4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$.

For $k=1$ we say that $\|\cdot\|$ is a $\tilde{F}$-norm. Moreover, if a $\tilde{F}$-quasi-norm satisfies

(F5) $\|\lambda x_n\| \to 0$ if $\|x_n\| \to 0$,

then $\|\cdot\|$ is called F-quasi-norm. When $k=1$, we obtain an F-quasi-norm (see [2]). The sets V_ε of all x with $\|x\|<\varepsilon$ form a base of neighbourhoods of 0 for the topology, determined by $\tilde{F}$-quasi-norm.

2.1. *The topology, defined by the $\tilde{F}$-quasi-norm, is compatible with the vector space operations, i. e. λx and $x+y$ are continuous in both variables together.*

2.2. *Let X be a vector space over a field K with archimedean valuation. Let $\|\cdot\|$ be a $\tilde{F}$-quasi-norm on X. Then $\|\cdot\|$ is a F-quasi-norm.*

Proof. Given any $\lambda \in K$ and $x \in X$. By the definitions of archimedean valuation $|\cdot|$ it follows that, there exist "integers" $n \in K$ such that $n=e+e+\ldots+e$ (n times) and $|\lambda| \leqq |n|$. Then from (F3) we have $\|\lambda x\| \leqq \|nx\| \leqq (\Sigma_{i=1}^{n-2} k^i + 2k^{n-1})\|x\|$ and so the condition (F5) is satisfied.

2.3. *Let $\|\cdot\|: X \to \boldsymbol{R}^+$. We say that the space $(X, \|\cdot\|)$ has property* (B), *if*

(B) *for every $\lambda \in K$ there exists $m>0$ such that $\|\lambda x\| \leqq m\|x\|$ for every $x \in X$.*

From proof 2.2 it follows:

2.4. *Let X be a vector space over a field K with archimedean valuation. Let $\|\cdot\|$ be a $\tilde{F}$-quasi-norm on X. Then the space $(X, \|\cdot\|)$ has the property* (B).

In Section 5 we give an example of $\tilde{F}$-norm on a vector space over the field K with discrete valuation, not satisfying (F5). In the same section is also given an example of F-norm, not satisfying (B).

3. Given a vector space over a field K with nontrivial valuation $|\cdot|$. The functional ρ with values in $\boldsymbol{R}^+$, defined on X, will be called a m o d u l a r on X, if it satisfies the following conditions:

(M1) if $\rho(\alpha x)=0$ for every $\alpha \in K^*$ then $x=0$ and $\rho(0)=0$;

(M2) $\rho(\alpha x+\beta y) \leqq \rho(x)+\rho(y)$ for any vectors $x, y \in X$ and every $\alpha, \beta \in K$ with $|\alpha|+|\beta| \leqq 1$.

Let ρ be a modular in X. The set $X_\rho=\{x: \lim_{\lambda\to 0} \rho(\lambda x)=0, x \in X\}$ will be called a m o d u l a r s p a c e.

It is easy to prove

3.1. *If ρ is a modular in X, then the set X_ρ is a vector subspace of the space X.*

3.2. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and ρ be a modular on X. Then the functional $|x|_\rho=\inf\{|u|>0: \rho(x/u) < |u|\}$ is an $\tilde{F}$-quasi-norm in X_ρ with $k=r=\inf\{|u|>1: u \in K\}$.*

Proof. For $x \in X_\rho$ the set $\{|u|>0 : \rho(x/u) < |u|\}$ is nonempty, so $|x|_\rho \in [0, \infty)$.

(F1) If $|x|_\rho = 0$, then $\rho(\alpha x/u) \leq |u|/|\alpha|$ for all α, $u \in K^*$. Hence, $\rho(\alpha x) \leq \rho(\alpha x/u) \leq |u|/|\alpha|$ for any $0<|u|<1$. Now we choose a sequence of elements $u_n \in K$ such that $|u_n| \to 0$. Then $\rho(\alpha x) \leq |u_n|/|\alpha|$ implies $\rho(\alpha x) = 0$ for every $\alpha \in K^*$. Hence by (M1) we have $x=0$. Converse implication is obvious.

(F2) Let $\lambda \in K$ and $|\lambda| \leq 1$. Then because $\rho(\lambda x/u) \leq \rho(x/u)$ we have $|\lambda x|_\rho = \inf\{|u|>0 : \rho(\lambda x/u) \leq |u|\} \leq \inf\{|u|>0 : \rho(x/u) \leq |u|\} = |x|_\rho$.

(F3) Now we shall prove the triangle inequality (F3). Let $x, y \in X_\rho$ consider two cases:

a) K is not discretely valued, so $|K|$ is dense in $\boldsymbol{R}^+$. Given any $\varepsilon>0$ by density, there exist elements $u, v, w \in K$ such that

$$|x|_\rho + \varepsilon/5 \leq |u| \leq |x|_\rho + 2\varepsilon/5, \quad |y|_\rho + \varepsilon/5 \leq |v| \leq |y|_\rho + 2\varepsilon/5,$$
$$|x|_\rho + |y|_\rho + 4\varepsilon/5 \leq |w| \leq |x|_\rho + |y|_\rho + \varepsilon.$$

Now we observe that $|u/w| + |v/w| = (|u|+|v|)/|w| \leq (|x|_\rho + |y|_\rho + 4\varepsilon/5)/(|x|_\rho + |y|_\rho + 4\varepsilon/5)$ and by (M1) we have

$$\rho((x+y)/w) = \rho(ux/wu + vy/wv) \leq \rho(x/u) + \rho(y/v) \leq |u| + |v|$$
$$\leq |x|_\rho + |y|_\rho + 4\varepsilon/5 \leq |w|.$$

Hence $|x+y|_\rho \leq |x|_\rho + |y|_\rho + \varepsilon$ and the triangle inequality follows.

b) K is discretely valued and $r>1$ is the generator of the value group of K, i. e. $|K^*| = \{r^n : n \in \boldsymbol{Z}\}$. Because 0 is the only limit point of $|K^*|$, hence $|X_\rho| \subset |K|$. We may assume $x \neq 0$ and $y \neq 0$, since otherwise (F3) is obvious. Then $|x|_\rho = |v| = r^k$, $|y|_\rho = |u| = r^l$, for some $u, v \in K$ and $k, l \in \boldsymbol{Z}$.

Now we consider two cases:

(i) there exist $w \in K$ and $m \in \boldsymbol{Z}$ such that $|x|_\rho + |y|_\rho = |w| = r^m$. But $|u/w| + |v/w| = (r^l + r^k) r^{-m} = 1$. Hence, by (M1)

$$\rho((x+y/w) \leq \rho(ux/wu + vy/wv) \leq \rho(x/u) + \rho(y/v) \leq |u| + |v| = |w| = |x|_\rho + |y|_\rho$$

So $|x+y|_\rho \leq |x|_\rho + |y|_\rho$.

(ii) Suppose $w \in K$, $m \in \boldsymbol{Z}$ be chosen such that $|w| = r^m$ and $r^m \leq |x|_\rho + |y|_\rho < r^{m+1}$. If $|x|_\rho + |y|_\rho = r^m$, then we have the condition (i). Hence let $r^m < |x|_\rho + |y|_\rho < r^{m+1}$. Thus $\rho((x+y)/\alpha_0 w) = \rho(ux/\alpha_0 wu + vy/\alpha_0 wv)$, where $\alpha_0 \in K$ and $|\alpha_0| = r$. But $|u/\alpha_0 w| + |v/\alpha_0 w| = (r^k + r^l) r^{-m-1} < r^{m+1}\, r^{-m-1} = 1$. So we have $\rho((x+y)/\alpha_0 w) \leq |\alpha_0 w| = r^{m+1} < r(|x|_\rho + |y|_\rho)$. It follows $|x+y|_\rho \leq r(|x|_\rho + |y|_\rho)$.

(F4) Let $x \in X_\rho$ and $\lambda_n \to 0$. Given any $\alpha \in K$, $\alpha \neq 0$. We have by (M2) $\rho(\lambda_n x/\alpha) \to 0$. Hence $\rho(\lambda_n x/\alpha) \leq |\alpha|$ for sufficiently large index n. But 0 is the limit point of $|K^*|$. So, it finally implies $|\lambda_n x|_\rho \to 0$.

In Section 5 we give an example of the modular ρ such that in the triangle inequality for $|\cdot|_\rho$ we have $r>1$, and for $r=1$ the triangle inequality is false.

3.3. *Let ρ be a modular on X and let $\lambda, \gamma \in K$ be such that $|\lambda|>1$, $|\gamma|<1$ and $|\lambda^2\gamma|<1$. Then for every $x \in X_\rho$ we have $|\gamma|\,|\lambda x|_\rho \leq |x|_\rho$.*

Proof. Let $y \in X_\rho$,

$$\{|u|>0 : \rho(\frac{\lambda}{\lambda\gamma}\,\frac{\lambda\gamma y}{u}) \leq |u|\} \subset \{|u|>0 : \rho(\frac{\lambda}{\lambda\gamma}\,\frac{\lambda\gamma y}{u}) \leq |\frac{u}{\lambda\gamma}|\}$$
$$= \{|\lambda\gamma|\,|\frac{u}{\lambda\gamma}| : \rho(\frac{\lambda}{\lambda\gamma}\,\frac{\lambda\gamma y}{u}) \leq |\frac{u}{\lambda\gamma}|\}.$$

Now we observe that $|\gamma||\lambda y|_\rho=|\gamma|\inf\{|u|>0:\rho(\lambda y/u)\leq|u|\}=|\gamma|\inf\{|u|>0:\rho(\lambda y\lambda\gamma/\lambda\gamma u\leq|u|\}$. Hence $|\gamma||\lambda y|_\rho\geq|\gamma||\lambda\gamma||\gamma^{-1}y|_\rho$, it implies $|\lambda y|_\rho\geq|\lambda\gamma||y/\gamma|_\rho$.

Now let $x\in X_\rho$ and we denote $y=\lambda\gamma x$. Thus $|\lambda^2\gamma x|_\rho\geq|\lambda||\gamma||y/\gamma|_\rho=|\lambda||\gamma||\lambda x|_\rho$, it follows $|\gamma||\lambda x|_\rho\leq|\lambda|^{-1}|\lambda^2\gamma x|_\rho\leq|\lambda^2\gamma x|_\rho\leq|x|_\rho$.

3.4. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and let ρ be a modular on X. Then the space $(X_\rho,|\cdot|_\rho)$ has the property* (B).

Proof. Suppose that the condition (B) is not satisfied. Then there exists $\lambda\in K^*$ such that for every $m\in\boldsymbol{N}$ there exists $x\in X_\rho$ such that $|\lambda x|_\rho>m|x|_\rho$. Now we choose $m\in\boldsymbol{N}$, $\beta\in K$ such that $|\lambda|<m$, $|\beta|<m$ and $|\lambda^2/\beta|<1$. By $\sim$(B) there exists an element $x_0\in X_\rho$ such that $|\lambda x_0|_\rho>m|x_0|_\rho>|\beta||x_0|_\rho$. Hence

$$(*)\qquad |\lambda x_0|_\rho>|\beta||x_0|_\rho.$$

We denote $\gamma=e/\beta$, then $|\gamma|<1$ and $|\lambda^2\gamma|<1$. So by 3.2. it follows $|\beta|^{-1}|\lambda x_0|_\rho\leq|x_0|_\rho$ and we get a contradiction with (*). From 3.2 and 3.4. it follows

3.5. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and ρ be a modular on X. Then the functional $|\cdot|_\rho$ is an F-quasi-norm in X_ρ.*

So we have

3.6. *If ρ is a modular on X, then with the topology, defined by the $|\cdot|_\rho$, a space $(X_\rho,|\cdot|_\rho)$ is a topological vector space.*

3.7. *If ρ is a modular on X, then $|x|_\rho<1$ implies $\rho(x)\leq|x|_\rho$.*

Proof. Let $|x|_\rho<1$. For $x\neq0$ there exists $u\in K^*$ such that $|x|_\rho\leq|u|<1$. So by the definition $|\cdot|_\rho$ we have $\rho(x/u)\leq|u|$. Hence $\rho(x)=\rho(ux/u)\leq\rho(x/u)\leq|u|$ and we obtain $\rho(x)\leq|x|_\rho$.

4. A modular ρ on X will be called convex whenever, if it satisfies the condition

(M3) $\rho(\alpha x+\beta y)\leq|\alpha|\rho(x)+|\beta|\rho(y)$ for any vectors $x,y\in X$ and $\alpha,\beta\in K$ such that $|\alpha|+|\beta|\leq1$.

4.1. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and ρ be a convex modular on X. Then the functional $\|x\|_\rho=\inf\{|u|>0:\rho(x/u)\leq1\}$ is a quasi-norm in X with $k=r=\inf\{|u|>1:u\in K\}$. In the case, when K is non-discretely valued, then $r=1$ and $\|\cdot\|_\rho$ is a norm.*

Proof. 4.1 is similar to 3.2. In the next section we give an example of a convex modular ρ such that in the triangle inequality for $\|\cdot\|_\rho$ we have $k>1$, and for $k_0=1$ the triangle inequality is false.

5.1. Let $K=Q_2$ and $X=Q_2\times Q_2$. We define for $x=(a,b)\in X$ $\rho(x)=|a|/(1+|a|)+31|b|/10(1+|b|)$, ρ is a modular on X, but ρ is not convex. In this case $|K^*|=\{2^n:n\in\boldsymbol{Z}\}$ and so $k=2$ in the triangle inequality $|\cdot|_\rho$.

Now we take $x_0=(2,0)$ and $y_0=(0,2)$. Then $|x_0|_\rho=\inf\{|u|>0:|u|^2+2|u|-2\geq0\}$, $|y_0|_\rho=\inf\{|u|>0:|u|^2+2|u|-6{,}2\geq0\}$, $|x_0+y_0|_\rho=\inf\{|u|>0:|u|^2+2|u|-8{,}2\geq0\}$. We consider the equations

$$t^2-2t-2=0,\quad t^2-2t-6{,}2=0,\quad t^2-2t-8{,}2=0.$$

Now denote by t_i the positive solution of equation (i). We obtain $0<t_1<1$, $1<t_2<2$, $2<t_3<3$. Hence $|x_0|_\rho=1$, $|y_0|_\rho=2$, $|x_0+y_0|_\rho=4$ and we have $|x_0+y_0|_\rho>|x_0|_\rho+|y_0|_\rho$.

5.2. Let $K=Q_2$, $X=Q_2\times Q_2$. If we define for $x=(a,b)\in X$ $\rho(x)=|a|+|b|$, then ρ is the convex modular on X. Hence by 4.1, $\|\cdot\|_\rho$ is a quasinorm with $k=2$. Given $x_0=(0,b_0)$, $y_0=(a_0,0)$ such that $|b_0|=1$, $|a_0|=2$. Thus for $x\in X$ $\|x\|_\rho=\inf\{|u|>0:|a|+|b|\leq|u|\}$ and so $\|x_0\|_\rho=\inf\{|u|>0:|u|\geq 1\}$, $\|y_0\|_\rho=\inf\{|u|>0:|u|\geq 2\}$, $\|x_0+y_0\|_\rho=\inf\{|u|>0:|u|\geq 3\}$. Hence $\|x_0\|_\rho=1$, $\|y_0\|_\rho=2$, $\|x_0+y_0\|_\rho=4$ and we obtain $\|x_0+y_0\|_\rho>\|x_0\|_\rho+\|y_0\|_\rho$.

5.3. Let K be nonarchimedean valued. Let X be a set of all infinite sequences of elements in K such that only a finite set of elements is different from zero. X becomes a vector space, when we define $x+y=(x_n+y_n)$ and αx, $\alpha\in K$, by $\alpha x=(\alpha x_n)$.

Let $x=(x_n)\in X$. Denote by S_x the support of the sequence x, i.e. $S_x=\{n\in \boldsymbol{N}: x_n\neq 0\}$. Now introduce a functional $\|\cdot\|$ on X by $\|x\|=\Sigma_{n\in S_x}|x_n|\max\limits^{n}(1,|x_n|)$. We use the convention that summation over an empty set gives (the origin of $\boldsymbol{R}$) 0. We show that $\|\cdot\|$ is an F-norm on X. Conditions (F1) and (F2) follow immediately from the definition $\|\cdot\|$.

(F3) Let $x,y\in X$. We consider the case

(i) $S_x\cap S_y=\emptyset$. Then $\|x+y\|=\|x\|+\|y\|$.

(ii) $S_x\cap S_y\neq\emptyset$. Thus

$$\|x+y\|=\sum_{n\in S_{x+y}}|(x+y)_n|\max\limits^{n}(1,|(x+y)_n|)$$

$$=\sum_{n\in S_x\setminus S_y}|x_n|\max\limits^{n}(1,|x_n|)+\sum_{n\in S_x\cap S_y}|x_n+y_n|\max\limits^{n}(1,|x_n+y_n|)$$

$$+\sum_{n\in S_y\setminus S_x}|y_n|\max\limits^{n}(1,|y_n|)$$

$$\leq\sum_{n\in S_x\setminus S_y}|x_n|\max\limits^{n}(1,|x_n|)+\sum_{n\in S_x\cap S_y}\max[|x_n|\max\limits^{n}(1,|x_n|),|y_n|\max\limits^{n}(1,|y_n|)]$$

$$+\sum_{n\in S_y\setminus S_x}|y_n|\max\limits^{n}(1,|y_n|)$$

$$\leq\sum_{n\in S_x\setminus S_y}|x_n|\max\limits^{n}(1,|x_n|)+\sum_{n\in S_x\cap S_y}|x_n|\max\limits^{n}(1,|x_n|)+\sum_{n\in S_y\setminus S_x}|y_n|\max\limits^{n}(1,|y_n|)$$

$$+\sum_{n\in S_x\cap S_y}|y_n|\max\limits^{n}(1,|y_n|)$$

$=\|x\|+\|y\|$. Hence $\|x+y\|\leq\|x\|+\|y\|$.

(F4) Let $x\in X$ and $\lambda_k\to 0$. Given ε, $0<\varepsilon<1$, there exists $k_0\in\boldsymbol{N}$ such that $|\lambda_k|\Sigma_{n\in S_x}|x_n|<\varepsilon$ for all $k\geq k_0$. Then

$$\|\lambda_k x\|=\sum_{n\in S_x}|\lambda_k||x_n|\max\limits^{n}(1,|\lambda_k||x_n|)\leq|\lambda_k|\sum_{n\in S_x}|x_n|<\varepsilon$$

for every $k\geq k_0$. Hence we obtain (F4).

(F5) Let $\| x(k) \| \to 0$, $\lambda \in K$, where $x(k) \in X$ for every $k \in N$. We observe that

$$\lim_{k\to\infty} \sum_{n \in S_{x(k)}} | x(k)_n | \max^{n} (1, | x(k)_n |) = 0.$$

Hence for sufficiently large integers k we have $| x(k)_n | < 1$ for all $n \in S_{x(k)}$. Then for sufficiently large k

$$(**) \qquad \| x(k) \| = \sum_{n \in S_{x(k)}} | x(k)_n |.$$

Now, given any ε, $0<\varepsilon<1$, by $(**)$ there exists $k_0 \in N$ such that $|\lambda| \Sigma_{n \in S_{x(k)}} | x(k)_n | < \varepsilon$. It follows that

$$\| \lambda x(k) \| = \sum_{n \in S_{x(k)}} | \lambda | \, | x(k)_n | \max^{n} (1, | \lambda | \, | x(k)_n |) = | \lambda | \sum_{n \in S_{x(k)}} | x(k)_n | < \varepsilon$$

for all $k \geq k_0$. And finally (F5) is true.

Now we shall show that the space $(X, \| \cdot \|)$ does not satisfy the (B) property. Suppose, that (B) holds in $(X, \| \cdot \|)$, i. e. for every $\lambda \in K$ there exists $m>0$ such that $\| \lambda x \| \leq m \| x \|$ for every $x \in X$. Given $\lambda_0 \in K$, $|\lambda_0| > 2$, consider the sequence

$$x(l) = \{\underbrace{e, \ldots, e}_{l \text{ times}}, 0, \ldots, \}, \quad l = 1, 2, \ldots .$$

Then $\| x(l) \| = l$, $\| \lambda_0 x(l) \| = \Sigma_{n=1}^{l} | \lambda_0 |^{n+1}$ and by (B) we obtain $\Sigma_{n=1}^{l} 2^n \leq \Sigma_{n=1}^{l} | \lambda_0 |^{n+1} \leq ml$. Hence $(2^l - 1)/l \leq m/4$ for all $l \in N$. We obtain a contradiction and so the properties (B) are not true.

5.4. Let K and X be the same as in 5.3. We define $\| x \| = \Sigma_{n \in S_x} [\, | x_n | \max (1, | x_n |)]^n$.

The proof of the conditions (F1) to (F4) is similar as in 5.3. But we show that the condition (F5) in this case is false.

Given $\lambda \in K$, $| \lambda | > 1$, and consider a sequence $x(l) = \{0, \ldots, 0, e/\lambda, 0, \ldots\}$, $l = 1, 2, \ldots$, such that $S_{x(l)} = \{l\}$. Then we have $\| \lambda x(l) \| = 1$, but $\| x(l) \| = 1/| \lambda |^l \to 0$ as $l \to \infty$. Hence (F5) is not true. It implies that the condition (B) is not satisfied in X.

We observe that in 5.3 and 5.4 the mappings $\| \cdot \|$ were examples of modulars, not satisfying the condition (B).

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