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GENERIC GÂTEAUX DIFFERENTIABILITY OF LOCALLY LIPSCHITZIAN FUNCTIONS

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Summary. In 1973 Kenderov [2] proved the following result: Theorem (Kenderov). Let X be a separable Banach space and $\varphi: X \to \boldsymbol{R}$ be a locally Lipschitzian function such that $\lim_{t\to 0_+}(\varphi(x+th)-\varphi(x))/t$ exists for each $x \in X$ and each h, belonging to a dense subset of X. Then φ is Gâteaux differentiable on some second category subset of X.

It is shown in the paper that analogous result holds not only for separable Banach spaces, but for the introduced by Ekeland, Lebourg [1] H-spaces, too. A Banach space is named H-space, if it admits a nonnegative Fréchet differentiable function with bounded and nonempty support. Every reflexive Banach space is an H-space.

By involving generic Gâteaux differentiability properties of locally Lipschitzian functions it is possible to obtain the recent result of Konyagin [3] due to a construction of Lau, which partially answers a Stechkin's conjecture.

In 1963 Stechkin [7] conjectured that any metric projection mapping P_M, generated by an arbitrary subset M of a strictly convex Banach space, has to be with non-multivalued images for the points of a second category subset of the space.

That conjecture has been confirmed for separable strictly convex Banach spaces (Konyagin [4], Zajiček [9]) and strictly convex H-spaces (Lau [5], Konyagin [4]).

In 1919 H. Rademacher [6] proved that any locally Lipschitzian mapping from an open subset of $\boldsymbol{R}^m$ into $\boldsymbol{R}^n$ has to be differentiable almost everywhere on its domain. It is a natural question to ask whether the same is true where 'almost everywhere' is understood not in the sense of Lebesgue measure, but in the sense of Baire category. Unfortunately the answer is negative. There are known examples of Lipschitzian functions, defined on intervals of the real line, which fail to be differentiable on dense G_δ subsets. It is interesting to see at what additional conditions the set of points of differentiability of a locally Lipschitzian function turns out to be a second category set.

A result of that kind was obtained in 1973 by P. Kenderov.

Theorem (Kenderov [2]). *Let X be a separable Banach space and $\varphi: X \to \boldsymbol{R}$ be a locally Lipschitzian functional such that* $\lim_{t\to 0_+}(\varphi(x+th)-\varphi(x))/t$ *exists for each $x \in X$ and each h belonging to a dense countable subset Γ of X. Then φ is generically Gâteaux differentiable.*

The aim of this paper is to show that an analogous result holds for another class of Banach spaces — the so-called H-spaces introduced by

Ekeland and Lebourg [1]. A Banach space is an H-space, if it admits nonnegative Fréchet differentiable function with bounded and nonempty support. It is known that a reflexive Banach space is an H-space, because it admits an equivalent Fréchet differentiable norm [8].

As an application the recent result of S. Konyagin [4] due to a construction of K. S. Lau [5] partially answering a Stechkin's hypothesis has been obtained.

The Stechkin's hypothesis: Let X be a strictly convex Banach space and P_M be a metric projection generated by some subset $M \subset X$. Then the set of points, at which P_M is not multivalued, is a second category subset of X.

The Stechkin's hypothesis has been confirmed for separable strictly convex Banach spaces (Konyagin [3], Zajiček [8]) and strictly convex H-spaces (Lau [5], Konyagin [4]).

Definition 1 (Ekeland and Lebourg [1]). Let X be a Banach space and $F: X \to \boldsymbol{R}$ be a function. The bounded linear functional $x^* \in X^*$ is ε-supporting F at $x \in X$ if there is $\delta > 0$ such that $F(x+h) - F(x) \geqq \langle x^*, h \rangle - \varepsilon \| h \|$, whenever $\| h \| < \delta$.

The set of all ε-supporting F functionals at x is named ε-support of F at x and is denoted by $S_\varepsilon F(x)$.

Theorem 2 (Ekeland and Lebourg [1]). *Let X be an H-space and $F: X \to \boldsymbol{R}$ be a lower semicontunuous function. Then for each $\varepsilon > 0$ there exists a dense subset $D_\varepsilon \subset X$ such that $S_\varepsilon F(x) \neq \emptyset$ whenever $x \in D_\varepsilon$.*

Lemma 3. *Let $\varphi: X \to \boldsymbol{R}$ be a locally Lipschitzian function and $\varepsilon > 0$ be fixed Denote $T_\gamma(\varphi) = \{x \in X: \exists\, x^* \in S_\varepsilon \varphi(x)$ such that $\delta > \gamma\}$. Then $S_\varepsilon \varphi(x_0) \neq \emptyset$ whenever $x_0 \in \overline{T_\gamma(\varphi)}$.*

Proof. The assumptions imply the existence of sequences $(x_n) \subset T_\gamma$, $x_n \to x_0$ and $(x_n^*) \subset X^*$ such that

$$\varphi(x_n + h) - \varphi(x_n) \geqq \langle x_n^*, h \rangle - \varepsilon \| h \| \quad \text{for} \quad \| h \| < \gamma. \tag{1}$$

For some $\eta > 0$ whenever $x, y \in B(x_0, \eta)$ the inequality

$$| \varphi(x) - \varphi(y) | \leqq L(x_0) \| x - y \| \text{ holds.} \tag{2}$$

Let $\tau = \frac{1}{2} \min(\gamma, \eta)$, then

$$-\varepsilon + \langle x_n^*, \frac{h}{\| h \|} \rangle \leqq \frac{\varphi(x_n + h) - \varphi(x_n)}{\| h \|} \leqq L(x_0) \tag{3}$$

whenever $x_n \in B(x_0, \tau)$ and $\| h \| < \tau$ for great n. It follows from (3) that $\| x_n^* \| \leqq L(x_0) + \varepsilon$. According to Banach-Alaoglu theorem there is a weakly* convergent generalized subsequence $x_{\nu(\alpha)}^* \xrightarrow[\alpha \in A]{w^*} x_0^*$, $\| x_0^* \| \leqq L(x_0) + \varepsilon$.

Let fix h, $\| h \| < \tau$, and choose $\alpha \in A$ such that $x_{\nu(\alpha)} \in B(x_0, \tau)$. From (1) and (2) we get

$$\begin{aligned} \langle x_{\nu(\alpha)}^*, h \rangle &\leqq \varphi(x_{\nu(\alpha)} + h) - \varphi(x_{\nu(\alpha)}) + \varepsilon \| h \| \\ &\leqq \varphi(x_0 + h) - \varphi(x_0) + 2L(x_0) \| x_{\nu(\alpha)} - x_0 \| + \varepsilon \| h \|. \end{aligned}$$

Finally, after transition, where $\alpha \in A$, it follows that $\langle x_0^*, h \rangle \leqq \varphi(x_0 + h) - \varphi(x_0) + \varepsilon \| h \|$, i. e. $x_0^* \in S_\varepsilon \varphi(x_0)$.

Lemma 4. *Let* $\varphi: X \to \mathbf{R}$ *be a locally Lipschitzian function, which is* ε-*supported on a dense subset. Then for some dense* G_δ *subset* $U \subset X$, *when* $x \in U$, *there is* $x^* \in X^*$ *such that*

$$\langle x^*, h\rangle \leq \overline{\lim_{t\to 0_+}} \frac{\varphi(x+th)-\varphi(x)}{t} + 2\varepsilon \text{ for each } h \in X,\ \|h\|=1.$$

Proof. Denote

$$D_\varepsilon = \{x \in X:\ S_\varepsilon\varphi(x) \neq \emptyset\}, \quad \bar{D}_\varepsilon = X,$$

$$T_n = \{x \in X:\ \exists\, x^* \in S_\varepsilon\varphi(x), \text{ with } \delta > 1/n\},$$

$$B_k T_n = \{z \in X:\ \|z-x\| < 1/kn, \quad x \in T_n\},$$

$$U_k = \bigcup_{n=1}^{\infty} B_k T_n, \quad U = \bigcap_{k=1}^{\infty} U_k.$$

Then $D_\varepsilon = \bigcup_{n=1}^{\infty} T_n$ and U is a second category set.

Let $x_0 \in U$ and $|\varphi(x)-\varphi(y)| \leq L(x_0)\|x-y\|$ for $x, y \in B(x_0, \eta)$. For arbitrarily taken integer k, $x_0 \in U_k$ and there is n_k, for which $x_0 \in B_k T_{n_k}$. There are two possibilities: The first one is that for any k it is possible to choose n_{k+1} to be greater than n_k and the second one is that for some fixed k and m it is true that $x_0 \notin \bigcup_{i=m}^{\infty} B_k T_i$. The latter implies $B(x_0, \frac{1}{ki}) \cap T_i = \emptyset$ for $i \geq m$, but then $B(x_0, \frac{1}{si}) \cap T_i = \emptyset$ for $i \geq m$ and $s \geq k$, i. e. $x_0 \notin \bigcup_{i=m}^{\infty} B_s T_i$ for $s \geq k$. This means that $x_0 \in \bigcap_{s \geq k} (\bigcup_{i=1}^{m-1} B_s T_i)$. Then there is j such that $x_0 \in \bar{T}_j$. From Lemma 3 it follows $S_\varepsilon\varphi(x_0) \neq \emptyset$ and then

$$\langle x_0^*, h\rangle \leq \overline{\lim_{t\to 0_+}} \frac{\varphi(x_0+th)-\varphi(x_0)}{t} + 2\varepsilon, \ \|h\|=1.$$

In the first case there exists a sequence $x_k \to x_0$ such that

i. $x_k \in T_{n_k}$, where $n_k < n_s$ for $k < s$,

ii. $\|x_0 - x_k\| < 1/kn_k$, i. e. $x_0 \in B_k T_{n_k}$.

The corresponding to (x_k) sequence $(x_k^*) \subset X^*$ satisfies (4)

$$\varphi(x_k+h)-\varphi(x_k) \geq \langle x_k^*, h\rangle - \varepsilon\|h\| \text{ for } \|h\| \leq 1/n_k \tag{4}$$

and as was shown above $\|x_k^*\| \leq L(x_0)+\varepsilon$.

To apply Banach-Alaoglu theorem to obtain a weakly* convergent generalized subsequence $x_{\nu(\alpha)}^* \xrightarrow[\alpha \in A]{w^*} x_0^*$, where A is a directed partially ordered set and $\|x_0^*\| \leq L(x_0)+\varepsilon$.

Let fix $h \in X$, $\|h\|=1$ and choose $\alpha \in A$ such that

$|\langle x_{\nu(\alpha)}^*, h\rangle - \langle x_0^*, h\rangle| < \varepsilon/3$, $1/\nu(\alpha) < \varepsilon/3L(x_0)$ and $x_{\nu(\alpha)} + t'h \in B(x_0, \eta)$, where $t' = 1/n_{\nu(\alpha)}$.

It follows from (4)

$$\begin{aligned}\langle x_0^*, t'h\rangle < \langle x_{\nu(\alpha)}^*, t'h\rangle + \varepsilon t'/3 &\leq \varphi(x_{\nu(\alpha)}+t'h) - \varphi(x_{\nu(\alpha)}) + \varepsilon t'/3 \\ &\leq \varphi(x_0+t'h) - \varphi(x_0) + 2\varepsilon t'.\end{aligned}$$

(5) $$\langle x_0^*, h\rangle < \frac{\varphi(x_0+t'h)-\varphi(x_0)}{t'} + 2\varepsilon.$$

Obviously (5) holds for infinitely many values of t, which might be chosen arbitrarily small, because $1/n_{\nu(\alpha)} \xrightarrow[\alpha \in A]{} 0$, and then

(6) $$\langle x_0^*, h\rangle \leq \overline{\lim_{t\to 0_+}} \frac{\varphi(x_0+th)-\varphi(x_0)}{t} + 2\varepsilon \text{ for } h \in X, \ \|h\|=1,$$

which completes the proof.

Lemma 5. *Let X be an H-space and $\varphi: X \to \boldsymbol{R}$ be a locally Lipschitzian function. Then there is a dense G_δ subset $G \subset X$ such that for each $x \in G$ exist $x^*, y^* \in X^*$; $\|x^*\|, \|y^*\| \leq L(x_0)$ and for each h, $\|h\|=1$ the following inequa lities hold:*

(7) $$\langle x^*, h\rangle \leq \overline{\lim_{t\to 0_+}} \frac{\varphi(x+th)-\varphi(x)}{t}$$

(8) $$\underline{\lim_{t\to 0_+}} \frac{\varphi(x+th)-\varphi(x)}{t} \leq \langle y^*, h\rangle.$$

Proof. As φ is locally Lipschitzian, it follows from Theorem 2 that for any $\varepsilon > 0$ φ is ε-supported on a dense set $D_\varepsilon \subset X$. For countably many values of ε arbitrarily close to 0 apply Lemma 4 for each ε value. There are to be countably many dense G_δ sets, at the points of which (6) is fulfilled. Their intersection is the set of points, at which (7) holds. Let proceed in the same way for $-\varphi$ to obtain (8) on some dense G_δ subset of X.

Theorem 6. *Let X be an H-space and $\varphi: X \to \boldsymbol{R}$ be a locally Lipschitzian function. Let Γ be a dense subset of X such that for each $x \in X$ and each $h \in \Gamma$ $\lim_{t\to 0_+}(\varphi(x+th)-\varphi(x))/t$ exists. Then φ is generically Gâteaux differentiable.*

Proof. Let $x \in G$ (from Lemma 5). It is a routine matter to verify that $\overline{\lim}_{t\to 0_+} t^{-1}(\varphi(x+th)-\varphi(x))$ exists for each $h \in X$ (for instance [2]). As $\overline{\lim}_{t\to 0_+} t^{-1}(\varphi(x+th)-\varphi(x))$ coincides with $\underline{\lim}_{t\to 0_+} t^{-1}(\varphi(x+th)-\varphi(x))$, then for some $x^*, y^* \in X^*$ it follows from Lemma 5 that $\langle x^*, h\rangle \leq \langle y^*, h\rangle$ and $\langle x^*, -h\rangle \leq \langle y^*, -h\rangle$, $\forall h$. Hence $\langle x^*, h\rangle = \langle y^*, h\rangle = \lim_{t\to 0_+} t^{-1}(\varphi(x+th)-\varphi(x))$.

The next result concerning points of non-multivaluedness of metric projections is also a corollary of Lemma 5.

Theorem 7 (Lau [5], Konyagin [4]). *Let X be a strictly convex H-space and $P_M: X \to M$ be a metric projection. Then the Stechkin's conjecture is true.*

Proof. Let $x \in G$ (from Lemma 5) and let $z_1, z_2 \in P_M x$. Denote $p(x) = \inf_{z \in M} \|x-z\|$. Then $\lim_{t\to 0_+}(p(x+th_i)-p(x))/t = -1$ where $h_i = (z_i - x)/\|z_i - x\|$, $i=1, 2$. For some $x^* \in X^*$ (7) is fulfilled and it is true that $\langle x^*, h_i\rangle = -1$. This means $1 \leq \overline{\lim}_{t\to 0_+}(p(x+t\tilde{h}_i)-p(x))/t$, where $\tilde{h}_i = -h_i$, $i=1, 2$. The last inequality, as was shown in [9], is a sufficient condition for z_1 to coincide with z_2 and thus non-multivaluedness of P_M at x is obtained.

REFERENCES

1. I. Ekeland, G. Lebourg. Generic Fréchet differentiability and perturbed optimization problems in Banach spaces. *Trans. Amer. Math. Soc.*, **224**, 1976, 193-216.
2. П. Кендеров. Квазидиференцируемите функционали са диференцируеми почти навсякъде. — В: Матем. и матем. образование '73. София, 1974, 123-126.
3. С. В. Конягин. Аппроксимативные свойства произвольных множеств в банаховых пространствах. *Доклады АН СССР*, **239**, 1978, № 2, 261-264.
4. С. В. Конягин. Аппроксимативные свойства замкнутых подмножеств в банаховых пространствах и характеризация строго выпуклых пространств. *Доклады АН СССР*, **251**, 1980, № 2, 276—279.
5. Ka-Sing Lau. Almost Chebyshev subsets in reflexive Banach spaces. *Indiana Univ. Math. J.*, **27**, 1978, No 5, 791-795.
6. H. Rademacher. Über partielle und totale Differenzierbarkeit. *Math. Ann.*, **79**, 1919, 340-359.
7. S. B. Stechkin. Approximative properties of Banach spaces subsets. *Rev. Roum. Math. Pures Appl.*, **8**, 1963, 5-8.
8. S. Trojanski. On locally convex and differentiable norm in certain non-separable Banach spaces. *Studia Math.*, **37**, 1971, 173-180.
9. L. Zajiček. On the points of multivaluedness of metric projections in separable Banach spaces. *Comment. math. Univ. Carol.*, **19**, 1978, No 3, 513-523.
10. N. V. Zhivkov. Continuity and non-multivaluedness properties of metric projections and antiprojections. *Serdica. Bulg. Math. Publ.*, **8**, 1982, 378-385.

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