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ON ABSOLUTE CONVERGENCE OF FOURIER TRANSFORMS.

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It's well known that Fourier transforms of the function f is as follows

(1) $$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt.$$

Absolute convergence of Fourier transforms namely the convergence of integral $\int_{-\infty}^{\infty} |\hat{f}(x)|\, dx$ is studied similarly to absolute convergence of Fourier series. One can find the theorems concerning absolute convergence of Fourier series in the fundamental monographs of N.K.Bari [1] and A.Zygmund [2].

In the paper of Możejko [3] the problem of the convergence of integral $\int_{-\infty}^{\infty} |x|^{\gamma} |\hat{f}(x)|^{\varepsilon} dx$, where $\gamma \geq 0$, $0 < \varepsilon < 2$ is studied. In the case of Fourier series similar problem is considered by A.A.Konushkov ([1],p.647) and for functions of bounded variation by Zygmund A.([2],p.384-388).

In the given note the problem of convergence of integral

(2) $$\int_{-\infty}^{\infty} \varphi^{2-\varepsilon}(x) |\hat{f}(x)|^{\varepsilon} dx,$$

where $0 < \varepsilon < 2$, φ is an even positive function determined along the ntire real axes satisfying the statement

(3) $$\int_{-\infty}^{\infty} \varphi^2(x)\, dx = +\infty$$

The problems of absolute convergence of almost ortogonal series are considered in the works [4] and [5].

Let $f_h(x) = f(x+h)$, $f_{-h}(x) = f(x-h)$,

$$\omega(\delta) = \omega(\delta, f) = \sup_{|h| \leq \delta} |f(x+h) - f(x-h)|,$$

$$\omega^{(r)}(\delta) = \omega^{(r)}(\delta, f) = \left\{ \sup_{|h| \leq \delta} \int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^r dx \right\}^{1/r},$$

where $r = 1, 2$.

According to Parceval equality we'll have

(4) $$\int_{-\infty}^{\infty} |f_h(x) - f_{-h}(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}_h(x) - \hat{f}_{-h}(x)|^2 dx = 4 \int_{-\infty}^{\infty} |\hat{f}(x)|^2 \sin^2 hx\, dx.$$

So, we'll have

(5) $$\int_{2^{n-1}}^{2^n} |\hat{f}(x)|^2 \sin^2 hx\, dx \leq \frac{1}{2} \left[\omega^{(2)}(2h) \right]^2.$$

Let $h=\pi 2^{-(n+1)}$. Then

$$(6)\quad \int_{2^{n-1}}^{2^n}|\hat{f}(x)|^2dx \le 2\int_{2^{n-1}}^{2^n}|\hat{f}(x)|^2\sin^2 hx\,dx \le \frac{1}{2}\left[\omega^{(2)}\left(\frac{\pi}{2^n}\right)\right]^2.$$

Lemma 1. Let $f\in L_2(-\infty,\infty)$, φ is the even positive function determined on the entire real axes and satisfying the statement (3). Let F,R and Q be positive functions satisfying the statement

$$(7)\qquad F(2^n)\int_{2^{n-1}}^{2^n}R(x)dx > \beta > 0,$$

$$(8)\qquad \int_c^\infty \varphi^2(x)\,F^{\frac{2}{2-\varepsilon}}(x)dx \le Q(c),$$

where $c\ge 2^{n-1}$, $0<\varepsilon<2$.

Then the inequality

$$(9)\quad \int_{2^{n-1}}^{2^n}\varphi^{2-\varepsilon}(x)|\hat{f}(x)|^\varepsilon dx \le \frac{2^{-\varepsilon/2}}{\beta}\int_{2^{n-1}}^{2^n}R(x)Q^{1-\varepsilon/2}(x)\left[\omega^{(2)}\left(\frac{\pi}{x}\right)\right]^\varepsilon dx$$

is true.

Proof. According to (7) we'll have

$$J_n=\int_{2^{n-1}}^{2^n}\varphi^{2-\varepsilon}(x)|\hat{f}(x)|^\varepsilon dx \le \frac{1}{\beta}\int_{2^{n-1}}^{2^n}\varphi^{2-\varepsilon}(x)|\hat{f}(x)|^\varepsilon F(x)\int_{2^{n-1}}^{x}R(y)dy\,dx$$

Changing the order of integrating and using the Hölders inequality to inner integral we'll find

$$J_n \le \frac{1}{\beta}\int_{2^{n-1}}^{2^n}R(y)\left(\int_y^{2^n}|\hat{f}(x)|^2dx\right)^{\varepsilon/2}\left(\int_y^{2^n}\varphi^2(x)F^{\frac{2}{2-\varepsilon}}(x)dx\right)^{1-\varepsilon/2}dy$$

Taking into account (6),(8) and the property of the modulus of continuity we'll have

$$J_n \le \frac{2^{-\varepsilon/2}}{\beta}\int_{2^{n-1}}^{2^n}R(x)Q^{1-\varepsilon/2}(x)\left[\omega^{(2)}\left(\frac{\pi}{x}\right)\right]^\varepsilon dx$$

Lemma 1 is proved.

As the convergence of series $\sum\limits_{n=1}^{\infty}R(n)Q^{1-\varepsilon/2}(n)\left[\omega^{(2)}\left(\frac{\pi}{n}\right)\right]^\varepsilon$

provides the convergence of integral

$$\int_1^\infty R(x)Q^{1-\varepsilon/2}(x)\left[\omega^{(2)}\left(\frac{\pi}{x}\right)\right]^\varepsilon dx$$

hence, on the base of lemma 1 we'll get the following statement.

Theorem 1. In the condition of lemma 1 the convergence of series

$$\sum_{n=1}^{\infty}R(n)Q^{1-\varepsilon/2}(n)\left[\omega^{(2)}\left(\frac{\pi}{n}\right)\right]^\varepsilon$$

provides the convergence of integral

$$\int_{-\infty}^{\infty}\varphi^{2-\varepsilon}(x)|\hat{f}(x)|^\varepsilon dx.$$

From this theorem one may derive various statement. In particular we'll have Theorem which contains the result of the work [3].

Theorem 2. Let $f\in L_2,\ 0<\varepsilon<2,\ \frac{\varepsilon}{2}-1\le\gamma<+\infty$

and series converge $\sum_{n=1}^{\infty} n^{\gamma-\varepsilon/2}\,[\omega^{(2)}(\frac{\pi}{n})]^{\varepsilon}$

Then the integral converges $\int_{-\infty}^{\infty}|x|^{\gamma}\,|\hat{f}(x)|^{\varepsilon}dx$.

Consequence. If for function $f\in L_2$ the series converge $\sum_{n=1}^{\infty} n^{-1/2}\,\omega^{(2)}(\frac{\pi}{n})$

then the integral $\int_{-\infty}^{\infty}|\hat{f}(x)|\,dx$ converges too.

Theorem 3. Let $f\in L_2,\ 0<\varepsilon<2,\ \frac{\varepsilon}{2}-1\le\gamma<\frac{\varepsilon}{2}$

and the series converge $\sum_{n=2}^{\infty} n^{-1}[\omega^{(2)}(\frac{\pi}{n})]^{\varepsilon}(\ln n)^{\gamma-\frac{\varepsilon}{2}+1}$

The the integral converges $\int_{2}^{\infty} x^{\varepsilon/2-1}\,|\hat{f}(x)|^{\varepsilon}\ln^{\gamma}x\,dx$.

Theorem 4. Let $f\in L_2,\ 0<\varepsilon<2,\ \frac{\varepsilon}{2}-1\le\gamma<\frac{\varepsilon}{2}$

and the series converge $\sum_{n=16}^{\infty} n^{-1}[\omega^{(2)}(\frac{\pi}{n})]^{\varepsilon}(\ln\ln n)^{\gamma-\frac{\varepsilon}{2}+1}$

Then the integral converges

$$\int_{16}^{\infty} x^{\varepsilon/2-1}\,|\hat{f}(x)|^{\varepsilon}(\ln x)^{\varepsilon/2-1}(\ln\ln x)^{\gamma}dx.$$

Lemma 2. Let the function of bounded variation $f\in L_1\cap L_2$, φ is an even positive function determinedon the entire real axes and satisfying the statement (3). Let F,R,Q be positive functions satisfying the statements (7) and (8).

Then at $0<\varepsilon<2$ inequality is true

$$\int_{2^{n-1}}^{2^n}\varphi^{2-\varepsilon}(x)\,|\hat{f}(x)|^{\varepsilon}dx\le\frac{2^{-\varepsilon/2}}{\beta}\int_{2^{n-1}}^{2^n}x^{-\frac{\varepsilon}{2}}R(x)Q^{1-\varepsilon/2}(x)[\omega^{(1)}(\tfrac{\pi}{x})]^{\varepsilon/2}dx.$$

Theorem 5. Taking statements of lemma 2 the covergence of series

$$\sum_{n=1}^{\infty} n^{-\varepsilon/2}R(n)Q^{1-\varepsilon/2}(n)[\omega^{(1)}(\tfrac{\pi}{n})]^{\varepsilon/2}$$

provides the convergence of integral

$$\int_{-\infty}^{\infty}\varphi^{2-\varepsilon}(x)\,|\hat{f}(x)|^{\varepsilon}dx.$$

Next theorem contains the results of the work [3].

Theorem 6. Let the function of bounded variation $f\in L_1\cap L_2$, $0<\varepsilon<2,\ \frac{\varepsilon}{2}-1\le\gamma<+\infty$ and the series converge

$$\sum_{n=1}^{\infty} n^{\gamma-\varepsilon}[\omega^{(1)}(\tfrac{\pi}{n})]^{\varepsilon/2}$$

Then the integral converges $\int_{-\infty}^{\infty}|x|^{\gamma}\,|\hat{f}(x)|^{\varepsilon}dx$.

Consequence. Let the function of bounded variation $f\in L_1\cap L_2$

and series converge $\sum_{n=1}^{\infty} n^{-1}\left[\omega^{(1)}\left(\frac{\pi}{n}\right)\right]^{1/2}$

Then the integral converges $\int_{-\infty}^{\infty} |\hat{f}(x)|\,dx.$

Theorem 7. Let the function of bounded variation $f \in L_1 \cap L_2$ $0 < \varepsilon < 2$, $\frac{\varepsilon}{2} - 1 \leq \gamma < \frac{\varepsilon}{2}$ and series converge

$$\sum_{n=2}^{\infty} n^{-\varepsilon/2 - 1}\left[\omega^{(1)}\left(\frac{\pi}{n}\right)\right]^{\varepsilon/2} (\ln n)^{\gamma - \frac{\varepsilon}{2} + 1}$$

Then the integral converges

$$\int_{2}^{\infty} x^{\varepsilon/2 - 1} |\hat{f}(x)|^{\varepsilon} (\ln x)^{\gamma}\,dx$$

Theorem 8. Let the function of bounded variation $f \in L_1 \cap L_2$,

$0 < \varepsilon < 2$, $\frac{\varepsilon}{2} - 1 \leq \gamma < \frac{\varepsilon}{2}$

and series converges

$$\sum_{n=16}^{\infty} n^{-\frac{\varepsilon}{2} - 1}\left[\omega^{(1)}\left(\frac{\pi}{n}\right)\right]^{\varepsilon/2} (\ln \ln n)^{\gamma - \frac{\varepsilon}{2} + 1}$$

Then the integral converges

$$\int_{16}^{\infty} (x \ln x)^{\varepsilon/2 - 1} |\hat{f}(x)|^{\varepsilon} (\ln \ln x)^{\gamma}\,dx.$$

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