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ERROR ESTIMATIONS FOR NUMERICAL SOLUTION OF ELLIPTIC EQUATIONS

L. G. Alexandrov

The purpose of this paper is to represent error estimations for numerical solution of an elliptic equation. The error will be estimated in Sobolev's discrete norms by means of average moduli of smoothness of the exact solution and its derivatives.

1. Average moduli of smoothness. Definitions and some properties.

Let R_n be the n-dimensional Euclidean space; Ω^n is the n-dimensional cube, $\Omega^n = \{x=(x_1,\dots,x_n) : 0 \leqq x_i \leqq 1, i=1,\dots,n\}$.
We confine to give our definitions in such domain for simplicity and because it is sufficient for our applications.
Let f be an integrable and bounded in Ω^n real function. The class of these functions we denote by $M(\Omega^n)$, i.e.
$M(\Omega^n) = \{ f: \longrightarrow R_1 : f \in L(\Omega^n), \sup_x |f(x)| < \infty, x \in \Omega^n \}$.
For every integer k and positive δ we define for $f \in M(\Omega^n)$:
def. 1. Local modulus of smoothness in the point $x \in \Omega^n$ by

$$\omega_k(f,x,\delta) = \sup\{ |\Delta_h^k f(t)| : t,\dots,t+kh \in \Omega^{n\,(x)}_{k\delta/2} \}$$

where $\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(t+mh)$

and $\Omega^{n\,(x)}_{k\delta/2} = \{ y \in \Omega^n : |x-y|_{R_n} \leqq k\delta/2 \}$;

def.2. Average modulus of smoothness (τ-modulus) by

$$\tau_k(f,\delta)_{L_p} = \|\omega_k(f,\cdot,\delta)\|_{L_p} = \left\{\int_{\Omega^n}|\omega_k(f,x,\delta)|^p dx\right\}^{1/p} .$$

These moduli were used first by Bl.Sendov and N.Korovkin in one-dimensional case and by V.Popov in many-dimensional case.

In our considerations we shall use some important properties of these moduli. These properties concern the dependence of τ-moduli on f and explain why the estimations involving τ-moduli are useful.

Properties:

P1. Let $f \in M(\Omega^n)$ and $p < \infty$. Then the modulus $\tau_1(f,\delta)_{L_p} = o(1)$ iff f is integrable in Riemann's sense ($f \in Rie(\Omega^n)$).

We use usual notations to denote the partial derivatives of the functions. For a multiindex $\alpha = (\alpha_1,\dots,\alpha_n)$, where α_i (i=1,...,n) are integers or zero and $|\alpha| = \alpha_1+\dots+\alpha_n$ we denote

$$D^{\alpha}f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}} .$$

P2. If for $|\alpha|=1$ $D^{\alpha}f \in M(\Omega^n)$, then for every positive δ and integer $k \geq 1$

$$\tau_k(f,\delta)_{L_p} \leq \delta \sum_{|\alpha|=1} \tau_{k-1}(D^{\alpha}f, k\delta/(k-1))_{L_p} .$$

P3. If for $0 \leq \alpha_i \leq 1$ (i=1,...,n) $D^{\alpha}f \in M(\Omega)$, then

$$\tau_1(f,\delta)_{L_p} \leq c \sum_{\substack{0\leq\alpha_i\leq 1\\ |\alpha|\geq 1}} \delta^{|\alpha|}\|D^{\alpha}f\|_{L_p} .$$

Note:

Here and throughout the rest c is an independent on f and δ constant.

P4. If $f \in W_p^k(\Omega^n)$ ($W_p^k(\Omega^n)$ is Sobolev's space) and $n/p < k$, then

$$\tau_k(f,\delta)_{L_p} \leq c\delta^k \sum_{|\alpha|=k} \|D f\|_{L_p}$$

The next property was proved only in the case when $n=2$.

P5. If f is continuous and possesses bounded variations (we follow A.S.Kronrod's definitions see [3]) $\mathrm{Var}_1(f,\Omega)$ and $\mathrm{Var}_2(f,\Omega)$, then

$$\tau_1(f,\delta)_{L_p} \leq c\left[\delta\,\mathrm{Var}_2(f,\Omega) + \delta^2\,\mathrm{Var}_1(f,\Omega)\right]^{1/p}\left[\omega(f,\delta)\right]^{1-1/p},$$

where $\omega(f,\delta)$ is the classical modulus of continuity.

2. Preliminary notes and notations.

We shall consider a boundary differential problem in the square Ω^2,

$$\Omega^2 = \left\{ x=(x_1,x_2) \; : \; 0 \leq x_i \leq 1 \; , \; i=1,2 \right\} \; .$$

We denote the boundary of Ω^2 by Γ .In fact the case when the domain is a rectangle whose edges are parallel to the co-ordinate axes is not more common than this we discuss.The problem will be approximated on a square mesh Ω^2_h with parameter $h=1/N$ (N is integer). The mesh Ω^2_h consists of two systems of parallel lines :

$$\begin{cases} l^1_m: \; x_1 = m/h \\ l^2_m: \; x_2 = m/h \end{cases} \qquad m=0,1,\dots,N.$$ The set of knots we denote by

$\omega_h = \omega = \left\{ x=x^{ij}=(ih,jh) \; : \; i,j=0,1,\dots,N \right\}$; the discrete boundary by $\gamma = \omega \cap \Gamma$; the set of internal knots of ω by $\mathring{\omega} = \omega \setminus \gamma$ .To denote the differences and discrete norms of the network functions defined on ω we follow A.A.Samarski see [7] .For a network function y we use the following notations:

$$y_{x_1} = y^{ij}_{x_1} = (y^{i+1\,j} - y^{ij})/h \quad ; \qquad y_{\bar{x}_1} = y^{ij}_{\bar{x}_1} = (y^{ij} - y^{i-1\,j})/h \quad ;$$

$$y_{x_2} = y^{ij}_{x_2} = (y^{i\,j+1} - y^{ij})/h \quad ; \qquad y_{\bar{x}_2} = y^{ij}_{\bar{x}_2} = (y^{ij} - y^{i\,j-1})/h \quad ;$$

or more generally by y_{x_α} we denote the numerical derivative coresponding to the multiindex $\alpha=(\alpha_1,\alpha_2)$;

$$\|y\|_{L_p} = \left\{ \sum_{x\in\mathring{\omega}} |y(x)|^p h^2 \right\}^{1/p} \quad ;$$

$$\|y\|_{W_p^\nu} = \|y\|_{L_p} + \sum_{|\alpha|=\nu} \|y_{x_\alpha}\|_{L_p} \qquad \text{where} \quad \nu = 0,1,2 \ ;$$

$$\|y\|_C = \max_{x \in \overset{\circ}{\omega}} |y(x)| \ .$$

3. Formulation of the problem and error estimations.

We consider the elliptic problem:

$$\text{(I)} \begin{cases} \dfrac{\partial}{\partial x_1}\left(k_1(x_1)\dfrac{\partial u}{\partial x_1}\right) + \dfrac{\partial}{\partial x_2}\left(k_2(x_2)\dfrac{\partial u}{\partial x_2}\right) - q(x)u = f(x) & x \in \Omega^2 \\ u(x) = 0 & x \in \Gamma \end{cases}$$

where $0 < c_0 < k_i(x_i) < c_1 < \infty$ ($i=1,2$ and c_0, c_1 are absolute constants), $q(x) \geqq 0$, $q \in L_\infty$ and $f \in L_p$.

In the general case, here the solution u is a distribution and it is possible that the symbol $u(x)$ has no sense. That is why we shall operate below with an integral average of u, e.g. see [5 p. 33], which we shall denote again with u.

We approximate this problem following R.D.Lazarov see [6].

in order to formulate the difference scheme we need two operators introduced by Tihonov and Samarski see [7], [8]:

$$T^{x_1}u(x) = (hv_{11}(x_1))^{-1}\int_{x_1-h}^{x_1} v_{11}(\xi)\, u(\xi, x_2)\, d\xi + (hv_{21}(x_1))^{-1}\int_{x_1}^{x_1+h} v_{21}(\xi)\, u(\xi, x_2)\, d\xi$$

where $\quad v_{11}(\xi) = \displaystyle\int_{x_1-h}^{\xi} (k_1(s))^{-1}\, ds \quad , \quad v_{21}(\xi) = \int_{\xi}^{x_1+h} (k_1(s))^{-1}\, ds \ ;$

$$T^{x_2}u(x) = (hv_{12}(x_2))^{-1}\int_{x_2-h}^{x_2} v_{21}(\xi)\, u(x_1, \xi)\, d\xi + (hv_{22}(x_2))^{-1}\int_{x_2}^{x_2+h} v_{22}(\xi)\, u(x_1, \xi)\, d\xi$$

where $\quad v_{12}(\xi) = \displaystyle\int_{x_2-h}^{x_2} (k_2(s))^{-1}\, ds \quad , \quad v_{22}(\xi) = \int_{\xi}^{x_2+h} (k_2(s))^{-1}\, ds \ .$

Using the discrete operators $\Lambda_i y = (a_i y_{\bar{x}_i})_{x_i}$, $i=1,2$ where $a_i(x_i) = h/v_{1i}(x_i)$ we state the difference scheme, as follows:

$$(II)\begin{cases} \Lambda y = T^{x_2}(1)\Lambda_1 y + T^{x_1}(1)\Lambda_2 y - T^{x_1}T^{x_2}(q)y = T^{x_1}T^{x_2}f & x \in \mathring{\omega} \\ y(x) = 0 & x \in \gamma \end{cases}$$

It is easy to see that the error of this numerical method (I - II) $z(x) = y(x) - u(x) \quad x \in \omega$ satisfies the equations:

$$\begin{cases} \Lambda z = \eta_0 + \Lambda_1\eta_1 + \Lambda_2\eta_2 & x \in \mathring{\omega} \\ z = 0 & x \in \gamma \end{cases}$$

where
$$\eta_0 = T^{x_1}T^{x_2}(q)\,u - T^{x_1}T^{x_2}(qu)$$
$$\eta_1 = T^{x_2}u - T^{x_2}(1)u$$
$$\eta_2 = T^{x_1}u - T^{x_1}(1)u \quad .$$

In these terms the following lemmas are valid

Lemma 1. (stability)

$$\|z\|_{L_2} \leqq c\left[\|\eta_0\|_{L_2} + \|\eta_1\|_{L_2} + \|\eta_2\|_{L_2}\right]$$

see [6] ;

Lemma 2. (stability) If $k_i(x_i) = 1$, $x_i \in [0,1]$ $i=1,2$ and $q(x)=\mathrm{const}$, then for integer $p > 1$ and $\nu = 0,1,2$ we have

$$\|z\|_{W_p^\nu} \quad c\left(\frac{p^2}{p-1}\right)^4 \sum_{i=0}^{2}\|\eta_i\|_{W_p^\nu}$$

see [9] ;

Lemma 3. (approximation) For the multiindex $\alpha = (\alpha_1, \alpha_2)$ $|\alpha| \leqq 2$ we have

$$\|\eta_{0\,x_\alpha}\|_{L_p} \leqq c\|q\|_{L_\infty}\left[\tau_2(D^\alpha u,h)_{L_p} + \sum_{i=1}^{2}\|k_{3-i}\|_{L_\infty}\tau_1(D^\alpha u,h)_{L_p}\,\tau_1(k_i,h)_{L_{\frac{p}{p-1}}}\right]$$

$$\|\eta_{i\,x_\alpha}\|_{L_p} \leqq c\left[\tau_2(D^\alpha u,h)_{L_p} + \tau_1(D^\alpha u,h)_{L_p}\,\tau_1(k_i,h)_{L_{\frac{p}{p-1}}}\right] \quad i=1,2 \quad .$$

From these three lemmas immediatly follows the next theorem:

Theorem 1. If u and y are solutions of (I) and (II) respectively, then

$$(3.1)\quad \|y - u\|_{L_2} \leqq c\left\{ \tau_2(u,h)_{L_2} + \tau_1(u,h)_{L_2}\left[\tau_1(k_1,h)_{L_2} + \tau_1(k_2,h)_{L_2}\right]\right\}$$

$$c = c(c, c, \|q\|_{L_\infty}) \quad ;$$

If also $k_i(x_i) = 1$, $x_i \in [0,1]$ $i=1,2$ and $q(x) = \text{const}$, then for integer $p \geq 1$ and $\nu = 0,1,2$

$$(3.2)\quad \|y - u\|_{W_p^\nu} \leqq c\left(\frac{p^2}{p-1}\right)^4\left[q\,\tau_2(u,h)_{L_p} + \sum_{|\alpha|=\nu} \tau_2(D^\alpha u,h)_{L_p}\right] \quad .$$

Combining Theorem 1. with the properties P1. - P5. we obtain a number of corollaries concerning the convergence of this method.

Corollaries from (3.2) : For integer $p \geq 1$ and $\nu = 0,1,2$

C1. If for $|\alpha| = \nu$ $D^\alpha u \in M(\Omega^2) \cap Rie(\Omega^2)$, then

$$\|y - u\|_{W_p^\nu} = o(1) \quad .$$

C2. If for $|\alpha| = \nu+1$ $D^\alpha u \in M(\Omega^2)$, then

$$\|y - u\|_{W_p^\nu} = O(h) \quad .$$

C2'. If for $|\alpha| = \nu+1$ $D^\alpha u \in M(\Omega^2)$ $Rie(\Omega^2)$, then

$$\|y - u\|_{W_p^\nu} = o(h) \quad .$$

C3. If $u \in W_p^{\nu+2}(\Omega^2)$, then

$$\|y - u\|_{W_p^\nu} \leqq ch^2 \sum_{|\alpha|=\nu+2} \|D^\alpha u\|_{L_p} \quad .$$

C5. If for $|\alpha| = \nu$ $D^\alpha u$ are continuous and possess bounded variations, then

$$\|y - u\|_{W_p^\nu} \leqq c\left(\frac{p^2}{p-1}\right)^4 h^{1/p} \sum_{|\alpha|=\nu} \omega(D^\alpha u,h)^{1-1/p} \quad .$$

C6. If for $|\alpha| = \nu + 1$ $D^{\alpha}u$ are continuous and possess bounded variations, then

$$\| y - u \|_{W_p^{\nu}} \leq c \left(\frac{p^2}{p-1} \right)^4 h^{1+1/p} \sum_{|\alpha| = \nu+1} \omega (D^{\alpha}u, h)^{1-1/p} .$$

The corollaries from (3.1) are similar.

Finally, these corollaries and embedding of W_p^{ν} in C $(2/p < \nu)$ imply corresponding results for uniform convergence of the numerical solution.

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