КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 113—118

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 113-118

THE DESCARTES' RULE AND THE ČEBYŠEV'S SYSTEMS

Vladimir Vladimirovič Aleksandrov

1. Some notations and definitions :

$R := (-\infty, +\infty)$ denotes the real line supplied with the classical algebraic, topological and order structures.

$\dot{R}$ and $\ddot{R}$ are two compactifications of R : $\dot{R} := R \cup \{*\infty\}$ and $\ddot{R} := \{-\infty\} \cup R \cup \{+\infty\} =: [-\infty, +\infty]$, where : $*\infty$, $-\infty$, $+\infty$ are three new "infinite" elements. It is well-known that the topology and to a certain extent some algebraic properties are extended from R onto $\dot{R}$ and $\ddot{R}$. Besides, $\ddot{R}$ is totally ordered and $\dot{R}$ is oriented and homeomorphic to any circumference.

Z is the set of all integers of R.

Let u and v be arbitrary elements of $\ddot{R}$. Then :

${}_uZ_v := \{k \in Z \mid k \geq u,\ k \leq v\}$. The set ${}_uZ_v$ is considered to be totally ordered by the relstion $<$. Obviously : ${}_{-\infty}Z_{+\infty} = Z$, ${}_3Z_2 = \emptyset$, ${}_1Z_1 = \{1\}$, etc.

${}^uR^v$ is the set of all maps $c : {}_uZ_v \ni k \longrightarrow c_k \in R$, or otherwise ${}^uR^v$ is the set of all ${}_uZ_v$-ordered families or ${}_uZ_v$-systems of real numbers $c := (c_k \in R \mid k \in {}_uZ_v)$.

${}^uO^v := (O_k = 0 \mid k \in {}_uZ_v) = (0,\dots,0) \in {}^uR^v$ is the zero element of ${}^uR^v$.

$({}^uR^v)^* := {}^uR^v \setminus \{{}^uO^v\}$.

${}^uA^v$ is the alternation operator on ${}^uR^v$, that is

${}^uA^v : {}^uR^v \ni c = (c_k \mid k \in {}_uZ_v) \longrightarrow {}^uA^v c := ((-1)^k . c_k \mid k \in {}_uZ_v) \in {}^uR^v$.

V_c is the set of indices $k \in {}_{u+1}Z_v$ for which a given $c \in {}^uR^v$ has a variation, that is the set of numbers $k \in {}_{u+1}Z_v$ for which : either $c_k . c_{k-1} < 0$ or $c_{k-1} = c_{k-2} = \dots = c_{k-m+1} = 0$ and $c_k . c_{k-m} < 0$, for a certain $m \in {}_2Z_{k-u}$.

$|M| :=$ Card M is the cardinal number (or the number of elements) of a given set M.

2. Some systems of functions :

Hereafter let be given : a number $n \in {}_1Z_{+\infty}$ and a set $\tilde{S}$ which is not supplied with any predetermined topology (it is convenient to suppose in addition that $|\tilde{S}| > n$) and let

(1) $\tilde{F} := (\tilde{f}_k : \tilde{S} \longrightarrow R \mid k \in {}_0Z_n)$

be a fixed ${}_0Z_n$-system (or a ${}_0Z_n$-ordered family) of real-valued functions defined on the set $\tilde{S}$. For every $c \in {}^oR^n$, set : $\tilde{F}_c : \tilde{S} \ni s \longrightarrow \tilde{F}_c(s) := \sum (c_k.\tilde{f}_k(s) \mid k \in {}_0Z_n) \in R$. Such a function $\tilde{F}_c$ is said to be a $\tilde{F}$-polynomial or more detailed a $\tilde{F}$-polynomial with a system c of coefficients.

$N(\tilde{F}_c;S_o) := \{s \in S_o \mid \tilde{F}_c(s) = 0\}$ is the set of all zeroes of the $\tilde{F}$-polynomial $\tilde{F}_c$ on a given set S_o, where $S_o \subset \tilde{S}$. Obviously $N(\tilde{F}_{({}^o0^n)};S_o) = S_o$ for every subset S_o of the set $\tilde{S}$.

The system (1) is said to be a Čebyšev (Tchebyscheff;Chebyshev) system or a T-system on a set S_1, $S_1 \subset \tilde{S}$, iff for every $c \in ({}^oR^n)^*$ the inequality $|N(\tilde{F}_c;S_1)| \leqq n$ holds true.

The system (1) is said to be a Markov (Markoff) system or a M-system on a set $S_2 \subset \tilde{S}$, iff for every $m \in {}_0Z_n$ and every $c \in ({}^oR^n)^*$ for which $c_k = 0$, when $k \in {}_{m+1}Z_n$, the inequality $|N(\tilde{F}_c;S_2)| \leqq m$ holds true.

The system (1) is said to be a variationally Descartes system or a vD-system on a set S_3, $S_3 \subset \tilde{S}$, iff for every $c \in ({}^oR^n)^*$ the inequality $|N(\tilde{F}_c;S_3)| \leqq |V_c|$ holds true.

The system (1) is said to be an alternationally Descartes system or an aD-system on a set S_4, $S_4 \subset \tilde{S}$, iff for every $c \in ({}^oR^n)^*$ the inequality $|N(\tilde{F}_c;S_4)| \leqq |V_{({}^oA^n c)}|$ holds true.

The system (1) is said to be a varionally strictly Descartes system or a vsD-system on a set S_5, $S_5 \subset \tilde{S}$, iff for every $c \in ({}^oR^n)^*$ $|V_c| - |N(\tilde{F}_c;S_5)|$ is a non-negative even number.

The system (1) is said to be an alternationally strictly Descartes system or an asD-system on a set S_6, $S_6 \subset \tilde{S}$, iff for every $c \in ({}^oR^n)^*$ $|V_{({}^oA^n c)}| - |N(\tilde{F}_c;S_6)|$ is a non-negative even number.

If there exist sets S_7, S_8 and S_9 for which : $S_7 \subset S_9$, $S_8 \subset S_9$, $S_9 \subset \tilde{S}$, $S_7 \cap S_8 = \emptyset$ and the set $S_9 \setminus (S_7 \cup S_8)$ contains at most two different elements and if the system (1) : ...

(I) ... is a vD-system on S_7 and is an aD-system on S_8, then the system (1) is said to be a Descartes or a D-system on S_9.

(II) ... is a vsD-system on S_7 and is an asD-system on S_8, then the system (1) is said to be a strictly Descartes or a sD-system on S_9.

From now on (from the type of a system) the symbol E will denote any one fixed symbol among the symbols : T, M, vD, aD, D, vsD, asD, sD ; and the symbol e will denote a or v.

The E-system (1) on some set S_1, $S_1 \subset \tilde{S}$, is said to be an $E_{s'}$-system on S_1, where $s' \in \tilde{S}$, iff $\tilde{f}_k(s')=0$ for $k \in {}_1Z_n$ and $\tilde{f}_o(s') \neq 0$.

Let the system (1) be an E-system on some set S_2, where $S_2 \subset \tilde{S}$ and the set S_2 is totally ordered by a given relation r. If for all $k \in {}_1Z_n$ the functions $\tilde{f}_k$ are strictly r-monotonous on the set S_2 (that is, $\tilde{f}_k(s_1) < \tilde{f}_k(s_2)$ every time when $s_1 \; r \; s_2$ for $s_1, s_2 \in S_2$ and $k \in {}_1Z_n$) and $\tilde{f}_o(s) \neq 0$ for any $s \in S_2$, then the system (1) is said to be an E^r-system on the set S_2.

If the system (1) is an $E_{s'}$-system on S_3 and at the same time an E^r-system on S_3 then (1) is said to be an $E^r_{s'}$-system on the set S_3, where $S_3 \subset \tilde{S}$, $s' \in \tilde{S}$ and r is a relation of total order on S_3.

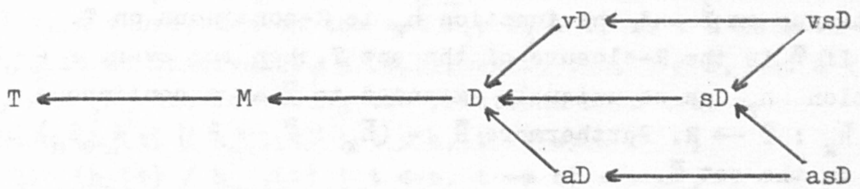
For an arbitrary number t', $t' \in R$, and for an arbitrary set T^o, $T^o \subset R$, it will denote $aT^o_{t'} := (-\infty, t') \cap T^o$ and $vT^o_{t'} := (t', +\infty) \cap T^o$.

3. Some Propositions :

In the terms introduced above, the following generalizations and complements to some results considered in [1] and [2] are established :

Proposition 1 : There are several maps of inclusion between the above-mentioned types of function systems.

1°. Some of them form the following diagram :



2°. Let $s' \in \tilde{S}$ and r be a relation of total order on a set S_o, $S_o \subset \tilde{S}$. Then similar diagrams are valid if all the symbols in the above diagram are supplied : ...

(I) ...only with upper indices r; (II) ...only with lower indices s'; (III) ...with upper indices r and lower indices s' at the same time.

Proposition 2 : For the system (1) and a certain matrix $J := (j_{km} \in R \mid k, m \in {}_oZ_n)$ the following function system can be formed

(2) $\quad J\tilde{F} := ((J\tilde{f})_k : S \longrightarrow R \mid k \in {}_oZ_n)$,

where $(J\tilde{f})_k(s) := \sum (j_{km} \cdot \tilde{f}_m(s) \mid m \in {}_oZ_n)$, for every $k \in {}_oZ_n$ and $s \in \tilde{S}$. In this case :

1°. If (1) is a T-system on a set S_1, $S_1 \subset \tilde{S}$, and J is an arbitrary non-degenerate matrix (that is, if $\det J \neq 0$), then (2) is a T-system on the set S_1.

2°. If (1) is a M-system on a set S_2, $S_2 \subset \tilde{S}$, and J is an arbitrary non-degenerate, lower-triangular matrix (that is, if :

det $J \neq 0$, $j_{kk} \neq 0$ and $j_{km} = 0$, for : $k \in {}_{o}Z_{n}$ and $m \in {}_{k+1}Z_{n}$), then (2) is a M-system on the set S_2.

3^o. If (1) is a D-system (an eD-system) on a set S_3, $S_3 \subset \tilde{S}$, and J is an arbitrary non-degenerate, non-negative, lower-triangular matrix (that is, if : det $J \neq 0$, $j_{kk} > 0$, $j_{kl} \geqq 0$ and $j_{km} = 0$, for : $k \in {}_{o}Z_{n}$, $l \in {}_{o}Z_{k-1}$ and $m \in {}_{k+1}Z_{n}$) then (2) is a D-system (an eD-system) on the set S_3.

4^o. If (1) is a sD-system (an esD-system) on a set S_4, $S_4 \subset \tilde{S}$, and J is a strictly-positive diagonal matrix (that is, if : $j_{kk} > 0$ and $j_{km} = 0$, for : $k \in {}_{o}Z_{n}$ and $m \in ({}_{o}Z_{k-1} \cup {}_{k+1}Z_{n})$, and consequently det $J \neq 0$), then (2) is a sD-system (an esD-system) on the set S_4.

Proposition 3 : Let be given : a number $n \in {}_{1}\dot{Z}_{+\infty}$, a set $\dot{S}$ and an E-system $\dot{F} := (\dot{f}_k : \longrightarrow R \mid k \in {}_{o}Z_{n})$ on the set $\dot{S}$.

1^o. Then a set $\dot{T}$, $\dot{T} \subset \dot{R}$, and a bijection $\dot{B} : \dot{S} \longrightarrow \dot{T}$ exist. Moreover, if $\dot{T} \neq \dot{R}$, then, without loss of generality, this bijection can be replaced by other (say,again $\dot{B}$), for which $* \infty \notin \dot{B}(\dot{S})$.

2^o. Let $T := \dot{T} \setminus \{*\infty\}$, $\dot{h}_k : \dot{T} \ni t \longrightarrow \dot{h}_k(t) := \dot{f}_k(\dot{B}^{-1}(t)) \in R$ and $h_k := \dot{h}_{k|T}$, for every $k \in {}_{o}Z_{n}$, then the systems $\dot{H} := (\dot{h}_k : \dot{T} \longrightarrow R \mid k \in {}_{o}Z_{n})$ and $H := (h_k : T \longrightarrow R \mid k \in {}_{o}Z_{n})$ are E-systems on the sets $\dot{T}$ and T respectively.

3^o. Under these conditions, for every $k \in {}_{o}Z_{n}$, the function $\dot{h}_k$ is $\dot{R}$-continous on $\dot{T}$ and the function h_k is R-continous on T.

4^o. If $\overline{T}$ is the R-closure of the set T, then for every $k \in {}_{o}Z_{n}$, the function h_k can be uniquely extended to $\overline{T}$ as a continuous function $\overline{h}_k : \overline{T} \longrightarrow R$. Furthermore $\overline{H} := (\overline{h}_k : \overline{T} \longrightarrow R \mid k \in {}_{o}Z_{n})$ is an E-system on the set $\overline{T}$.

5^o. Besides, there exist : a uniquely defined R-closed set $\underline{T}$ (that is, $\overline{\underline{T}} = \underline{T}$) and (maybe non-uniquely defined) an E-system $\underline{H} := (\underline{h}_k : \underline{T} \longrightarrow R \mid k \in {}_{o}Z_{n})$ on the set $\underline{T}$, such that : (I) $T \subset \underline{T} \subset R$; (II) $\underline{h}_{k|T} = h_k$ if $k \in {}_{o}Z_{n}$; (III) $\underline{T} \supset \underset{\sim}{T}$ every time, when $T \subset \underset{\sim}{T} \subset R$ and there is an E-system $\underset{\sim}{H} := (\underset{\sim}{h}_k : \underset{\sim}{T} \longrightarrow R \mid k \in {}_{o}Z_{n})$ on the set $\underset{\sim}{T}$, for which $\underset{\sim}{h}_{k|T} = h_k$ if $k \in {}_{o}Z_{n}$.

6^o. Moreover, if $\dot{F}$ is a T-system on $\dot{S}$, and $t' \in R$, then there exist such uniquely defind matrix J, for which the system $J\underline{H}$ is an $eD_{t'}$-system on the set $e\underline{T}_{t'}$, and a $D_{t'}$-system on the set $\underline{T}$.

7^o. Let $s^* := \dot{B}^{-1}(*\infty)$, $S := \dot{S} \setminus \{s^*\}$ and $B := \dot{B}_{|S}$. Then by means of the map $B^{-1} : T \longrightarrow S$, any structure (for example : the topology, the total order, etc.) which is defined on the set T can be transferred onto the set S.

8^o. More generally, there exist non-uniquely defined set $\underline{S}$ and a bijection $\underline{B} : \underline{S} \longrightarrow \underline{T}$, such that $\underline{S} \supset S$ and $\underline{B}_{|S} = B$. Then by means of the map $B^{-1} : \underline{T} \to \underline{S}$ every $\underline{T}$-structure can be transferred onto the set $\underline{S}$.

9^{o}. Finally, much of the constructions performed on the sets T, $\underline{T}$, etc. can be returned to the sets S, its extension $\underline{S}$ and their subsets. In particular, if $\underline{f}_k : \underline{S} \ni s \to \underline{f}_k(s) := \underline{h}_k(\underline{B}^{-1}(s)) \in R$, for every $k \in {}_oZ_n$, then $\underline{F} := (\underline{f}_k : \underline{S} \to R \mid k \in {}_oZ_n)$ is an E-system on the set $\underline{S}$ and $J\underline{F} := ((J\underline{f})_k : \underline{S} \to R \mid k \in {}_oZ_n)$ is a D-system (an eD-system) on the set $\underline{S}$ (on the set $B^{-1}e\underline{T}_{t'}$).

Remark 1 : In proposition 3 it is not supposed any predefined topology on the set S. This differs from the well-known assuption (cf. [1]), where a topology on the set S exists and much more S is compact with respect to it. Note also that the sets $\overline{T}$ and $\overline{\underline{T}}$ are R-closed, but they are not necessarily R-compact.

Proposition 4 : Let $n \in {}_1Z_{+\infty}$, T be a subset of R and $H := (h_k : T \to R \mid k \in {}_oZ_n)$ be a system of functions. A necessary and sufficient condition for the system H to be : ...

(I) ... a vD-system om some set T_1, $T_1 \subset T$, is : for every $m \in {}_1Z_n$ and for two arbitrarily chosen strictly increasing systems of numbers $(k_p \in {}_oZ_n \mid p \in {}_oZ_m)$ and $(t_q \in T_1 \mid q \in {}_oZ_m)$, where $m \in {}_1Z_n$, the inequality $\det (h_{k_p}(t_q) \mid p,q \in {}_oZ_m) > 0$ holds true.

(II) ...an aD-system on some set T_2, $T_2 \subset T$, is : the system $AH := ((-1)^k.h_k : T \to R \mid k \in {}_oZ_n)$ is a vD-system on the set T_2.

(III) ...a vsD-system on the set T_3, $T_3 \subset T$, is : H is a vD-system on T_3 ; T_3 is an interval $T_3 = (a,b)$, where : $a, b \in R$, and $a < b$; for every $m \in {}_1Z_n$ the following two relations hold true :

$$\lim (h_m(t) / h_{m-1}(t) \mid t > a, t \to a) = 0 ,$$
$$\lim (h_m(t) / h_{m-1}(t) \mid t < b, t \to b) = +\infty .$$

(IV) ...an asD-system on some set T_4, $T_4 \subset T$, is: the system AH is a vsD-system on the set T_4.

Remark 2 : The last proposition and the following simple example show that there is an error in [2], part 5, chapter 1, problem 41.

Example 1 : For every $t \in R$ it has : $t^3+t-10 = (t^2+2t+5).(t-2) =$

$$= \begin{cases} 1.t.(t+4).(t-3) - 1.t.(t+4) + 17.t - 10 ; \\ 1.t.(t+4).(t-2) - 2.t.(t+4) + 17.t - 10 ; \\ 1.t.(t+4).(t-1) - 3.t.(t+4) + 17.t - 10 . \end{cases}$$

In these three cases it has respectively :

$2 \notin (3,+\infty)$ and $|V| - |N| = 3 - 0 = 3$ is odd ;

$2 \notin (2,+\infty)$ and $|V| - |N| = 3 - 0 = 3$ is odd;

$2 \in (1,+\infty)$ and $|V| - |N| = 3 - 1 = 2$ is even.

Proposition 5 : Let be given : $n \in {}_1Z_{+\infty}$, $T \subset R$, $t' \in \hat{T}$ and a $D_{t'}$-system $H := (h_k : T \to R \mid k \in {}_oZ_n)$ on the set T, where $\hat{T}$ is the narrowest R-interval, which contain the set T.

1°. There exist an interval $\check{T} = (\check{a},\check{b})$, $T \subset \hat{T} \subset \check{T} \subset R$, and a system $W := (w_k : \check{T} \longrightarrow R \mid k \in {}_0Z_n)$ of functions non-decreasing on $\check{T}$ and strictly increasing on T.They generate the following function system $I := (i_k : \check{T} \longrightarrow R \mid k \in {}_0Z_n)$, where : $i_0(t) = 1$, for $t \in \check{T}$ and $i_k(t) := \int_{t'}^{t} dw_1(t_1) \int_{t'}^{t_1} dw_2(t_2) \ldots \int_{t'}^{t_{k-1}} dw_k(t_k)$ is an iterative (maybe improper) Stiltjes' integral for $k \in {}_1Z_n$ and $t \in \check{T}$. Furthermore $\lim (i_n(t) \mid t \longrightarrow \check{a}) = -\infty$ and $\lim (i_n(t) \mid t \longrightarrow \check{b}) = +\infty$.

2°. In these terms, for $k \in {}_0Z_n$ and $t \in T$ the representation $h_k(t) = h_o(t).i_k(t)$ holds true. Besides : I is a $D^{<}_{t'}$-system on T ; $eI := (i_k|_{eT_{t'}} \mid k \in {}_0Z_n)$ is an $eD^{<}_{t'}$-system on the set $eT_{t'}$; $eH := (h_k|_{eT_{t'}} \mid k \in {}_0Z_n)$ is an $eD_{t'}$-system on the set $eT_{t'}$;

3°. More over, if $\underline{T}$ is the largest subset of $\check{T}$ on which all the functions w_k, for $k \in {}_1Z_n$, are strictly increasing ; if the function $\underline{h}_o : \underline{T} \longrightarrow R \setminus \{0\}$ is a certain extension of the function $h_o : T \longrightarrow R$ (obviously $h_o(T) \subset R \setminus \{0\}$; and if for $k \in {}_1Z_n$, and $t \in \underline{T}$ one sets $\underline{h}_k(t) := \underline{h}_o(t).i_k(t)$, then $\underline{H} := (\underline{h}_k : \underline{T} \longrightarrow R \mid k \in {}_0Z_n)$ is a : $D_{t'}$-system on the set $\underline{T}$ and an $eD_{t'}$-system on the set $e\underline{T}_{t'}$.

4°. Let $aW_k := \lim(w_k(t) \mid t \in \underline{T},\ t\downarrow)$, $vW_k := \lim(w_k(t) \mid t \in \underline{T},\ t\uparrow)$ for every $k \in {}_1Z_n$. Then the system $\underline{H}$ is an $esD_{t'}$-system on the set $e\underline{T}_{t'}$ iff the set $e\underline{T}_{t'}$ is an interval, and in the case when $e\underline{T}_{t'} \neq \emptyset$ the equation $|eW_k| = +\infty$ holds true for every $k \in {}_1Z_n$.

5°. Conversely, if W is a system of functions which do not decrease on some R-interval $\check{T}$ and strictly increase on some subset T of $\check{T}$, and if $t' \in \check{T}$, then the system I generated by them is an $eD^{<}_{t'}$-system on $eT_{t'}$ and a $D^{<}_{t'}$-system on T. By that, if the system H is formed by means of multiplication of a system I with some function $h_o : T \longrightarrow R \setminus \{0\}$, then the system H is an $eD_{t'}$-system on $eT_{t'}$ and a $D_{t'}$-system on T .

Remark 3 : The present investigations can be applied to the generalized moment problem, to extremal values problems, and to other questions of the constructive theory of functions.

Refefences :

1. М.Г.Крейн ; А.А.Нудельман : Проблема моментов Маркова и экстремальные задачи, Москва, Наука, 1973.

2. G.Pólia ; G.Szegö : Aufgaben und Lehrsätze aus der Analysis, Band 2, Berlin, J.Springer, 1925 . (Teil 5, Kapitel 1).

Author's address : V.V.Aleksandrov, Institute of Mathematics, Bulgarian Academy of Sciences, P.O.Box 373, 1090 Sofia, Bulgaria.

Revised : 09.08.84.