SUPERCONVERGENCE OF THE FEM APPROXIMATIONS OF SOME PARABOLIC EQUATIONS

Andrey B. Andreev

1. Introduction. It is the purpose of the present paper to show that a superconvergence of the gradient for Galerkin-finite element metods estimate can be established in the case of the initial-boundary value problems for parabolic equations. We use (n^2+1)-point (n=1, 2) Lagrangian finite elements and numerical integration.

An error estimate in terms of a negative norm for the Galerkin method for parabolic equations has been investigated by Douglas, Dupont, Wheeler [6] ; Thomée [7] . In contrast with those papers, we obtain an error estimate of "superconvergence of the gradient" type. We would like point out the method, applied in the present paper, is an extension of that introduced by Zlamal [3] for elliptic problems (see also [4]).

2. Statement of the problem. Let Ω be a bounded domain in with a sufficiently smooth boundary Γ . We consider the following Cauchy problem for parabolic equation in a weak formulation [1,2] : find a function $u(x,t)\in \overset{0}{H}_1\times C_1[0,T]$ satisfying

(1) $\quad (\frac{\partial u}{\partial t},v)+a(u,v)= (f,v) \quad , \quad \forall\ v\in \overset{0}{H}_1(\Omega)\times C_1[0,T]$

and initial condition

(2) $\quad u(x,0)= u_0(x) \quad , \quad x\in\Omega \quad$ where

$$a(u,v)=\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x).\frac{\partial u}{\partial x_i}.\frac{\partial v}{\partial x_j}\,dx$$

3. Finite element method. In order to construct the finite element space S_n^h let us cover Ω by triangular elements. This partition we note by $\mathbb{K}$. Then

$$S_1^h=\{\varphi\in C(\Omega):\varphi=\alpha_1+\alpha_2x+\alpha_3y,\ (x,y)\in e,\ \forall\, e\in\ \mathbb{K},\ \alpha_j\in\mathbb{R},\ j=1,2,3\}$$

In this case the nodes of a given triangular element e coincide with

with the vertices a_j, j=1,2,3. For quadratic FE

$$S_2^h=\{\varphi \in C(\Omega): \varphi = \alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 xy+\alpha_5 x^2+\alpha_6 y^2, (x,y)\in e,\ e\in \mathbb{K},\ \alpha_i\in \mathbb{R},\ i=1,\dots,6\}$$

The nodes are vertices and the midpoints of the sides of the triangular elements. We consider the curved isoparametric elements. The triangle $\hat{e}=\{(\xi_1,\xi_2)\in \mathbb{R}^2: \xi_1=0, \xi_2=0, \xi_1+\xi_2\leqq 1\}$ will be called basic element. The theorem of superconvergence is true under condition that the partitions are 2-strongly regular (see [3] or [4]), i.e. there are constants $c_1,c_2>0$ such that

$$(3)\qquad |D^\alpha x_i^e|\leqq c_1 h_e^{|\alpha|},\quad |\alpha|\leqq 3,\ i=1,2,\ \forall e\in \mathbb{K}$$

$$c_2^{-1}h^2\leqq |J_e|\leqq c_2 h_e^2,$$

h_e=diam e, J_e is the Jacobian of the mapping

$$(4)\qquad \left|\begin{array}{l} x_1= x_1^e(\xi_1,\xi_2) \\ x_2= x_2^e(\xi_1,\xi_2). \end{array}\right.$$

We suppose that (4) is invertible.

Let the finite element partitions are such that

$$(5)\qquad \left| J_{e_1}^{-1}\frac{\partial x_i^{e_1}}{\partial \xi_1}\cdot\frac{\partial x_j^{e_1}}{\partial \xi_2} - J_{e_2}^{-1}\frac{\partial x_i^{e_2}}{\partial \xi_1}\cdot\frac{\partial x_j^{e_2}}{\partial \xi_2}\right|\leqq c.h,\quad i,j=1,2$$

for any two adjacent elements e_1,e_2. Obviously from (3) follows

$$c_3^{-1}h^2= \text{mes } e= c_3h^2\ ;\quad h=\max_e h_e$$

Let C be the generic constant. Denote by P(K) the class of polynomials of degree k.

We will use the quadrature formulas with positive coeficients exact for all polynomial from P(n+1), n=1,2

$$(6)\qquad I_e(f)\approx\iint_e f(x_1,x_2)dx\ ;\ \sum_{e\in K} I_e(f)\approx \iint_\Omega f(x_1,x_2)dx$$

We will try to find a solution u_h in the form $u_h(x,t) =\sum_j Q_j(t).\varphi_j(x)$ (semidiscrete Galerkin finite element method) where $\{\varphi_j(x)\}$ is a canonical basis in S_n^h.

Applying (6) for evaluation of the mass and stiffness matrices, we get for the approximate solutions of (1),(2)

$$(7)\qquad (\frac{\partial u_h}{\partial t},v)_h+\ a_h(u_h,v) = (f,v)_h$$

Let φ_I be the interpolant of the φ in S_n^h, i.e. $\varphi_I=\varphi$ in the nodes of the element $e\in K$.

We consider the seminorm:

$|v|_{1,h}^2=\sum_{e\in K} I_e((\frac{\partial v}{\partial x_1})^2+(\frac{\partial v}{\partial x_2})^2)$, which is a discret analog of

$$|v|^2_{1,\Omega}=\int_\Omega(\nabla v)^2 d\Omega$$

4. Some lemmas. Lemma 1. Let $I_e(\varphi)$ be any formula with positive coefficients which is exact for $\varphi\in P(n+1)$, n=1;2. Then $\{a_h(v,v)\}^{1/2}$ is a norm in S^h_n which is equivalent to the norm $|v|_{1,\Omega}$, i.e.

$$C^{-1}|v|^2_{1,\Omega}= a_h(v,v) = C|v|^2_{1,\Omega}$$

The elliptic problem corresponding to (7) is previously considered. For this purpose the following two lemmas are given.

Lemma 2. [5] Let the FE partitions be n-strongly regular and let the condition (5) be fulfilled. Finally, suppose that

(8) $$u\in H_{n+2}(\Omega);\quad a_{ij}(x)\in H_{n+1}(\Omega)$$
$$f(x)\in H_{n+1}(\Omega);\quad \left|\frac{\partial a_{ij}}{\partial x_k}\right|<\infty \qquad k=1,2;\ n=1,2.$$

Then

(9) $$|a_h(u-u_I,v)|\leq C.h^{n+1}\|u\|_{n+2,\Omega}|v|_{1,\Omega}\ ,\quad n=1,2$$

Using arguments similar to thos from [3], one can prove the following

Lemma 3. With the assumptions of lemma 2 we have

(10) $$|(f,v)-(f,v)_h|\leq C.h^{n+1}\|f\|_{n+1,\Omega}.|v|_{1,\Omega}$$

(11) $$|a(u,v)-a_h(u,v)|\leq C.h^{n+1}\|u\|_{n+2,\Omega}.|v|_{1,\Omega}\ ,\quad n=1,2$$

On $\hat e$ consider the following seminorms:

a) in the linear case

$$|\hat v|^*_{1,h,\hat e}=\left\{\frac{\text{mes } \hat e}{2}\left[\left(\frac{\partial\hat v}{\partial\xi_1}\right)^2(1/2,0)+\left(\frac{\partial\hat v}{\partial\xi_2}\right)^2(0,1/2)+\left(\frac{\partial\hat v}{\partial 1}\right)^2(1/2,1/2)\right]\right\}^{1/2}$$

b) in the quadratic case

$$|\hat v|^*_{1,h,\hat e}=\left\{\frac{\text{mes } \hat e}{4}\sum_{k=1}^{2}\left[\left(\frac{\partial\hat v}{\partial\xi_1}\right)^2(g_k,0)+\left(\frac{\partial\hat v}{\partial\xi_2}\right)^2(0,g_k)+\left(\frac{\partial\hat v}{\partial 1}\right)^2(g_k,g_{3-k})\right]\right\}^{1/2}$$

where $1=\sqrt{2}/2(1,-1)$ and $g_{1,2}=\frac{1}{2}(1\pm\frac{1}{\sqrt{3}})$ are the Gaussian points of $[0,1]$

For each $e\in K$ define $|v|^*_{1,h,e}$ by means of (4). Finally

$$|v|^*_{1,h}=\left(\sum_{e\in K}|v|^{*2}_{1,h,e}\right)^{1/2}$$ is a seminorm in S^h_n.

Lemma 4. For every $v\in S^h_n$ the following inequality holds:

(12) $$|v|_{1,h}\geq C.|v|^*_{1,h}$$

We define the semidiscrete norm $|||\cdot|||_{1,h}$ in the following way:

$$|||w(x,t)|||^2_{1,h}=\max_{t\in[0,T]}\|w\|^2_{L^2_h}+\int_0^T|w(x,t)|^2_{1,h}dt=\|w\|^2_{L_\infty(L^2_h)}+\|w\|^2_{L_2(H^1_h)}$$

where $\|\cdot\|_{L^2_h}$ is the discrete L_2-norm.

Lemma 5. Let $u(x,t)$ and $u_h(x,t)$ be the solutions of (1),(2) and (7) respectively. Let the conditions (8) of the lemma 2 be fulfilled for $\forall t\in[0,T]$, then

(13) $$|||u_h-u_I|||_{1,h}\leq C.h^{n+1}\left(\|u\|_{L_2(H_{n+2}(\Omega))}+\left\|\frac{\partial u}{\partial t}\right\|_{L_2(H_{n+1}(\Omega))}+\|f\|_{L_2(H_{n+1}(\Omega))}\right)$$

Proof: Set $z_h = u_I - u_h$, where u_I is the interpolant of the solution $u(x,t)$, $\forall t \in [0,T]$.

For $\forall v \in S_n^h$; $\forall t \in [0,T]$ we have

(14) $$(\frac{\partial z}{\partial t}h, v)_h + a_h(z_h, v) = (\frac{\partial u}{\partial t}h, v)_h - (\frac{\partial u}{\partial t}I, v)_h + (f,v)_h - (\frac{\partial u}{\partial t}h, v)_h$$
$$-a_h(u_I, v) = (f,v)_h - (f,v) + a(u,v) + (\frac{\partial u}{\partial t}, v) - (\frac{\partial u}{\partial t}I, v)_h - a_h(u_I, v)$$
$$+a_h(u,v) - a_h(u,v) = \{(f,v)_h - (f,v)\} + \{a(u,v) - a_h(u,v)\}$$
$$+\{a_h(u-u_I, v)\} + \{(\frac{\partial u}{\partial t}, v) - (\frac{\partial u}{\partial t}I, v)_h\}$$

Let us estimate every difference in the braces. Using (10),(11) and (9) we estimate the first three terms. The last term we present in the form

$$(\frac{\partial u}{\partial t}, v) - (\frac{\partial u}{\partial t}I, v)_h = (\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t}I, v) + (\frac{\partial u}{\partial t}I, v) - (\frac{\partial u}{\partial t}I, v)_h$$

Denote $w = \frac{\partial u}{\partial t}$ $\forall t \in [0,T]$, then

$$\int_e (w - w_I).v\, dx = \text{mes } e \int_{\hat{e}} (\hat{w} - \hat{w}_I).\hat{v}\, d\xi$$
$$\leq C.\text{mes } e.h^n \|w\|_{n+1,e} (\text{mes } e)^{-1/2} \|v\|_{o,e}$$

By summation over all $e \in K$ and applying Fridrich's inequality [2] we get

(15) $$\left|(\frac{\partial}{\partial t}(u-u_I), v)\right| \leq C.h^{n+1} \|\frac{\partial u}{\partial t}\|_{n+1,\Omega} |v|_{1,\Omega}$$

Let E be the error of a quadrature formula. If the quadrature formula is exact for all polynomial from P(n+1), according to the Bramble-Hilbert lemma [8] we have

$$|E_e(\sigma)| = \text{mes } e |E_{\hat{e}}(\hat{\sigma})| \leq C.\text{mes } e |\hat{\sigma}|_{n+2,\hat{e}} \quad , \quad \hat{\sigma} \in H_{n+2}(\hat{e})$$

Let $\hat{\sigma} = \frac{\partial \hat{u}}{\partial t}I.\hat{v}$. Obvious ly $E_e(\sigma) = 0$ for the case of the linear finite elements. Using Leibnitz formula for the quadratic one, we get

$$|E_e(\sigma)| \leq C.\text{mes } e \left|\frac{\partial \hat{u}}{\partial t}I\right|_{2,\hat{e}} |\hat{v}|_{2,\hat{e}} = C.\text{mes } e \left|\frac{\partial \hat{u}}{\partial t}I\right| |V|_{1,\hat{e}} \quad \text{(see [3] p668)}$$

Now: $$\left|\frac{\partial \hat{u}}{\partial t}I\right|_{2,\hat{e}} \leq \left|\frac{\partial(\hat{u}-\hat{u}_I)}{\partial t}\right|_{2,\hat{e}} + \left|\frac{\partial \hat{u}}{\partial t}\right|_{2,\hat{e}} \leq C\left|\frac{\partial \hat{u}}{\partial t}\right|_{3,\hat{e}} + \left|\frac{\partial \hat{u}}{\partial t}\right|_{2,\hat{e}} = Ch\|\frac{\partial u}{\partial t}\|_{3,e}$$

$|\hat{v}|_{1,\hat{e}} \leq C|v|_{1,e}$. Finally

(16) $$\left|(\frac{\partial u}{\partial t}I, v) - (\frac{u}{t}I, v)_h\right| \leq Ch^{n+1} \|\frac{\partial u}{\partial t}\|_{n+1,\Omega} |v|_{1,\Omega}$$

Return on the formula (14) with $v = z_h \in S_n^h$ and remark that

$$(\frac{\partial z}{\partial t}h, z_h) = \frac{1}{2}.\frac{\partial}{\partial t}\| z_h\|^2$$

Since the seminorms $|v|_{1,\Omega}$ and $|v|_{1,h}$ are equivalent from (9), (10),(11),(15) and (16) we obtain:

(17) $$\frac{1}{2}\frac{\partial}{\partial t}\|z_h\|^2_{o,h} + |z_h|^2_{1,h} = CM^2/2.h^{2(n+1)} + 1/2.|z_h|^2_{1,h}$$

where $M=\|u\|_{n+2,\Omega}+\|\frac{\partial u}{\partial t}\|_{n+1,\Omega}+\|f\|_{n+1,\Omega}$

The inequality (17) holds for every $t\in[0,T]$. Thus one can get the estimation (13) by integrating on t.

Remark. For the basic element we consider the quadrature formula

$$\int_{\hat{e}} f(\xi)d\xi \approx \frac{1}{6}\left[f(0,0)+f(1,0)+f(0,1)\right] \text{ -exact for } f\in P(1).$$

Applying this formula for n=1 to evaluate the mass matrix, we obtain discrete finite element schemes with a lumped mass matrix. The estimate (16) also holds:

$$\left|E_e\left(\frac{\partial u}{\partial t}I.v\right)\right| \leq C.\text{mes } e\left|\frac{\partial \hat{u}}{\partial t}I\right|_{1,\hat{e}}|\hat{v}|_{1,\hat{e}}=Ch^2\left\|\frac{\partial u}{\partial t}\right\|_{2,e}|v|_{1,e}.$$

5. The superconvergence result. We consider the seminorm $|\cdot|^*_{1,h,t}=\int_0^t|\cdot|^*_{1,h}\,dt$. We will prove the following main result:

Theorem. Let the conditions (8) of lemma 2 be fulfilled $\forall t\in[0,T]$. Then for the parabolic equation we have the estimate:

$$(18)\qquad |u-u_h|^*_{1,h,t}\leq Ch^{n+1}\left(\|u\|_{L_2(H_{n+2}(\Omega))}+\left\|\frac{\partial u}{\partial t}\right\|_{L_2(H_{n+1}(\Omega))}+\|f\|_{L_2(H_{n+1}(\Omega))}\right)$$

Proof: Let $w=u-u_I$. The proof deals separately with the two cases- the linear (n=1) and the quadratic (n=2) ones.

a) n=1

Consider the linear functionals: $l_1^1(\hat{u})=\frac{\partial\hat{w}}{\partial\xi_1}(1/2,0)$,

$l_2^1(\hat{u})=\frac{\partial\hat{w}}{\partial\xi_2}(0,1/2)$; $l_3^1(\hat{u})=\frac{\partial\hat{w}}{\partial l}(1/2,1/2)$ $\qquad l=\frac{1}{\sqrt{2}}(1,-1)$

They are continuous and bounded in $H_3(\hat{e})$

By simple calculations we get:

$$l_i^1(\hat{u})=0,\quad \hat{u}\in P(2),\ i=1,2,3.$$

Using Bramble-Hilbert lemma and summing over all $e\in\mathbb{K}$ we have:

$$(19)\qquad |u-u_I|^*_{1,h}\leq Ch^2\|u\|_{3,\Omega}$$

b) n=2

$$l_1^2(\hat{u})=\sum_{k=1}^{2}\left(\frac{\partial\hat{w}}{\partial\xi_1}\right)(g_k,0);\qquad l_2^2(\hat{u})=\sum_{k=1}^{2}\left(\frac{\partial\hat{w}}{\partial\xi_2}\right)(0,g_k);$$

$$l_3^2(\hat{u})=\sum_{k=1}^{2}\left(\frac{\partial\hat{w}}{\partial l}\right)(g_k,g_{3-k})\qquad l=\frac{1}{\sqrt{2}}(1;-1)$$

This functionals are continuous and bounded in $H_4(\hat{e})$. It is easy to show that $l_i^2(\hat{u})=0$ $\hat{u}\in P(3)$ $i=1,2,3$.

Then $|l_i^2(u)|\leq Ch^3\|u\|_{4,e}$ and summing over all $e\in\mathbb{K}$:

$$(20)\qquad |u-u_I|^*_{1,h}\leq Ch^3\|u\|_{4,\Omega}$$

Now integrating on t according to (19) and (20) we have

$$|u-u_I|^*_{1,h,t} \leq Ch^{n+1}\|u\|_{L_2(H_{n+2}(\Omega))} \quad , n=1,2$$

Hence, by (12) and (13) as well as by the triangle inequality we obtain:

$$|u-u_h|^*_{1,h,t} \leq |u-u_I|^*_{1,h,t} + |||u_I-u_h|||_{1,h}$$

which complete the proof of (18).

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Higher Institute of Machine and Electrical Engineering 5300-Gabrovo Bulgaria