КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 125—127

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 125-127

TWO-SIDED APPROXIMATION OF THE SOLUTION OF THE INITIAL PROBLEM FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS INVOLVING INEXACT DATA

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<u>1.Formulation of the problem.</u> Consider a system of n ordinary differential equations with initial conditions:

$$(1)\qquad \begin{cases} \dot{x} = f(t,x), \\ x(t_0) = x_0, \qquad t \in [t_0, t_0+\beta] \end{cases}$$

where inexact data for f and x_0 are involved, that is given two vector functions $\underline{f}(t,x)$, $\overline{f}(t,x)$ and two vectors $\underline{x}_0$, $\overline{x}_0$, find the set $\{\hat{x}\}$ of solutions $\hat{x}$ of (1) whenever f and x_0 satisfy:

$$(2)\qquad \begin{cases} \underline{f}(t,x) \le f(t,x) \le \overline{f}(t,x) \\ \underline{x}_0 \le x_0 \le \overline{x}_0 \end{cases}$$

under the assumptions that:

the functions $\underline{f}$, $\overline{f}$ are

(3_1) continuous with respect to t, Lipschitzian with resp.to x and
(3_2) quasi-monotone increasing with respect to x

in $[t_0, t_0+\beta] \times D$, $x_0 \in D \subset R^n$.

Our aim is:

i) to characterize the set $\{\hat{x}\}$;
ii) to give an estimation for the two-sided approximation of the solution set $\{\hat{x}\}$ by means of polygonal curves.

<u>2.Characterization of the solution set.</u> In order to characterize the solution set $\{\hat{x}\}$ we demonstrate that the above problem is reduced to the following pair of "end-point" problems:

$$(4)\qquad \begin{cases} \dot{x} = \underline{f}(t,x), \\ x(t_0) = \underline{x}_0, \end{cases}$$

$$(5)\qquad \begin{cases} \dot{x} = \overline{f}(t,x), \\ x(t_0) = \overline{x}_0. \end{cases}$$

Under assumptions (3_1) there exist solutions $\underline{\hat{x}}$, $\hat{\overline{x}}$ of (4), resp. (5) in some interval $[t_0, t_0+\alpha]$, $0<\alpha<\beta$.

It has been shown by M.Müller (cf. [1], [2], [4]) that the requirement (3_2) presents a sufficient condition for the problem considered to be of monotone type in the sense of L.Collatz.

We recall the definition of a quasi-monotone increasing function.

Definition. A vector function $f(t,x) = (f_1(t,x),\dots,f_n(t,x)$, defined on $D(f)$ depending on $x=(x_1,\dots,x_n)$ and possibly also on other variables combined in the symbol t, is said to be quasi-monotone incresing with respect to x, if for $i=1,\dots,n$ the inequality $f_i(t,x) \leq f_i(t,y)$ holds true whenever $x \leq y$, $x_i=y_i$, $(t,x),(t,y) \in D(f)$.

Under the assumption (3_2) for quasi-monotonicity of $\underline{f}$, $\overline{f}$ it can be proved that for the closed hull of $\{\hat{x}\}$ defined by

$$\operatorname{hull}\{\hat{x}\} = [\inf_{x\in\{\hat{x}\}} x(t),\ \sup_{x\in\{\hat{x}\}} x(t)]$$

we have

Theorem 1. $\operatorname{hull}\{\hat{x}\} = [\underline{\hat{x}}(t), \hat{\overline{x}}(t)]$.

The above theorem shows that problem (1)-(3) involving inexact data is reduced to problems (4), (5) involving exact data.

3. Two-sided approximation of the solution set. As an approximation tool we consider the class S_h of all continuous vector functions $s\colon [t_0, t_0+\alpha] \to R^n$, which components s_i are piece-wise linear functions with knots $\{t_\kappa\}_{\kappa=1}^{m}$, $t_\kappa=t_0+\kappa h$.

Theorem 2. There exist functions $\underline{s}=(\underline{s}_1,\dots,\underline{s}_n)$, $\overline{s}=(\overline{s}_1,\dots,\overline{s}_n) \in S_h$, such that

$$(i)\qquad [\underline{\hat{x}}, \hat{\overline{x}}] \subset [\underline{s}, \overline{s}]\ ,$$

$$(\mathrm{ii})\quad \begin{cases} \hat{\underline{x}}_i - \underline{s}_i \le n\alpha e^{n\alpha L}\,\omega(f,2Mh), \\ \bar{s}_i - \bar{x}_i \le n\alpha e^{n\alpha L}\,\omega(f,2Mh), \end{cases}$$

wherein $\omega(f,\delta) = \max_{\|x-\tilde{x}\|\le\delta,\ |t-\tilde{t}|\le\delta} \|f(t,x)-f(\tilde{t},\tilde{x})\|$, L is such that $|f_i(t,x_1,\ldots,x_n)-f_i(t,\tilde{x}_1,\ldots,\tilde{x}_n)| \le L\sum_{j=1}^{n}|x_j-\tilde{x}_j|$, and $M \ge 1/2$ is such that $|f(t,x)|\le M$ for $t\in[t_o, t_o+\alpha]$, $x\in D$.

The proof of Theorem 2 is constructive; we construct the approximating functions $\underline{s}$, $\bar{s}$ by means of a modification of Euler's method for systems of ordinary differential equation.

Finally we note that in the situation when i) the one-dimentional case (n=1) is considered, ii) no one-sideness of the approximation is required, and iii) exact input data ($\underline{f}=\bar{f}$, $\underline{x}_o=\bar{x}_o$) are assumed, an analogous estimate can be found in [3] (cf. Theorem 6.1 on p.179).

References

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