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A WHITNEY TYPE THEOREM FOR APPROXIMATING BY QUASIPOLYNOMIALS

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1. Let $\delta \subset \{1,\dots,n\} \stackrel{def}{=} \bar{n}$, $x \in R^n$, $x_\delta = \{x_i, i\in\delta\} \in R^{|\delta|}$, where $|\delta|$ - is the cardinality of δ. Similarly μ_δ and α_δ are defined, where

$$\mu_i,\ \alpha_i \in N;\quad \hat{x}_j = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n).$$

Similarly define $\hat{x}_\delta$. Further, $\delta \stackrel{def}{=} \{\delta(1),\dots,\delta(|\delta|)\}$,

$$\sum_{\alpha_\delta=0}^{\mu_\delta-1} = \sum_{\alpha_{\delta(1)}=0}^{\mu_{\delta(1)}-1} \cdots \sum_{\alpha_{\delta(|\delta|)}=0}^{\mu_{\delta(|\delta|)}-1};\quad x_\delta^{\alpha_\delta} = \prod_{i\in\delta} x_i^{\alpha_i}.$$

Denote $P_{\mu_\delta}(x)$ the polynomial of the group of variables x_δ of power $\mu_\delta - 1$ (i.e. of power $\mu_i - 1$ with respect to x_i, $i\in\delta$) with coefficients, depending on remaining variables:

$$P_{\mu_\delta}(x) = \sum_{\alpha_\delta=0}^{\mu_\delta-1} C_{\alpha_\delta}(\hat{x}_\delta)\, x_\delta^{\alpha_\delta}.$$

Let $\mathcal{D}$ denote some set of subsets $\bar{n}$; $\mu_{\mathcal{D}} \stackrel{def}{=} \{\mu_i \mid \exists \delta\in\mathcal{D}, \mu_i \in \mu_\delta\}$.
Expression

$$\prod_{\mu}^{\mathcal{D}}(x) = \sum_{\delta\in\mathcal{D}} P_{\mu_\delta}(x)$$

will be called quasipolynomial of power $\mu_{\mathcal{D}} - 1$.

Everywhere in this paper f - real or complex valued functions of n real variables. Let $K = I^n$, $I = [0,1]$, and $Y = Y(K)$ is one of the spaces $C(K)$ or $L_p(K)$, $0 < p \le \infty$. Given $\mathcal{D}$ and $\mu_{\mathcal{D}}$ consider the best approximation of function $f \in Y$ by set $(\prod_{\mu}^{\mathcal{D}})_Y$ of all quasipolynomials $\prod_{\mu}^{\mathcal{D}}(x) \in Y$:

$$E(f,\mathcal{D},\mu)_{Y(K)} = \inf_{\Pi_\mu^{\mathcal{D}}(x)\in(\Pi_\mu^{\mathcal{D}})_Y} \| f(x) - \Pi_\mu^{\mathcal{D}}(x)\|_{Y(K)} .$$

Let $X=X(T)$ -is a linear space of functions $f:T\to\mathcal{X}$, where $\mathcal{X}=R$ or C ; H - is a subspace of X. We shall call family of linear bounded operators $\nabla_\theta : X(T)\to X(T)$, depending on vectorial parameter $\theta\in G=G(H,T)$ exact annihilator of subspace H, if

$$f\in H \iff \forall\,\theta\in G ,\ \nabla_\theta f = 0.$$

In this paper an exact annihilator, caracterizing class $(\Pi_\mu^{\mathcal{D}})_Y$ is constructed. The power of the best approximation of function f by the set of quasipolynomials $(\Pi_\mu^{\mathcal{D}})_Y$ with the help of the exact annihilator of this set at spaces $C(K)$ or $L_p(K)$, $0<p\le\infty$ is established.

2. Everywhere in what follows s, s_k, i_s, j_s - natural numbers. Take

$$\eta\subset\mathcal{D},\ \bar\eta \overset{def}{=} \{\eta(s)\mid s\in\bar\eta\} = \{\eta(1),\dots,\eta(|\bar\eta|)\},\ \eta(s)\in\bar n ;\ x_\eta=\{x_s,\ s\in\bar\eta\}.$$

Similarly K_η is defined. Let $x_s^{(i_s)}\in I$, $1\le i_s\le\mu_s$, $s\in\bar{\mathcal{D}}$. We shall need notations

$$\sum_{k_\eta=1}^{\mu_\eta} \overset{def}{=} \sum_{k_{\eta(1)}=1}^{\mu_{\eta(1)}} \cdots \sum_{k_{\eta(|\bar\eta|)}=1}^{\mu_{\eta(|\bar\eta|)}} ;\quad |K_\eta| = \sum_{s\in\bar\eta} k_s ;$$

(1) $$\left(\prod x\right)_\eta \overset{def}{=} \prod_{s\in\bar\eta}\ \prod_{1\le i_s<j_s\le\mu_s} \left(x_s^{(i_s)} - x_s^{(j_s)}\right);$$

$$\left(\prod_{\ne k} x\right)_\eta \overset{def}{=} \prod_{s\in\bar\eta}\ \prod_{\substack{0\le i_s<j_s\le\mu_s\\ i_s,j_s\ne k_s}} \left(x_s^{(i_s)} - x_s^{(j_s)}\right);$$

$$f(\overset{\vee}{x}{}_j^{(1)}) \overset{def}{=} f(x_1,\dots,x_{j-1},x_j^{(1)},x_{j+1},\dots,x_n) .$$

$f(\overset{\vee}{x}^{(1)}_{\delta})(\delta\subset\bar{n})$ are defined similarly. Associate with set $\mathcal{D}$ family of operators

$$\underset{\mathcal{D}_{\mu,\theta}}{\nabla}:X(K)\longrightarrow X(K)\,,\quad r\overset{def}{=}\sum_{i\in\bar{\mathcal{D}}}\mu_i\,,$$

$$\theta=\left(x^{(1)}_{\mathcal{D}(1)},\ldots,x^{(\mu_1)}_{\mathcal{D}(1)},\ldots,x^{(1)}_{\mathcal{D}(|\bar{\mathcal{D}}|)},\ldots,x^{(\mu_{|\bar{\mathcal{D}}|})}_{\mathcal{D}(|\bar{\mathcal{D}}|)}\right)\in I^r;$$

(2) $$\underset{\mathcal{D}_{\mu,\theta}}{\nabla} f=\sum_{\eta\subset\mathcal{D}}\sum_{K_\eta=1}^{\mu_\eta}(-1)^{|K_\eta|-|\bar{\eta}|+|\eta|}f(\overset{\vee}{x}^{(K_{\bar{\eta}})}_{\bar{\eta}})\left(\prod_{\neq K} x\right)_{\eta}\left(\prod x\right)_{\bar{\mathcal{D}}\setminus\bar{\eta}}.$$

Associate with set $\eta=\emptyset$ in (1) number I, and with set in (2) - summand $f(x)(\prod x)_{\mathcal{D}}$. The next theorem shows, that the family of operators $\nabla_{\mathcal{D}_{\mu,\theta}}$ is exact annihilator of the class of quasipolynomials in spaces $Y(K)$.

T h e o r e m I. Let $f:K\to\mathcal{X}$, $f\in C(K)$ or $L_p(K)$, $0<p\leq\infty$.

Then

$$f(x)=\prod_{\mu}^{\mathcal{D}}(x)\iff\forall\theta\in I^r,\ \underset{\mathcal{D}_{\mu,\theta}}{\nabla} f=0.$$

Theorem I for functions $f\in C(K)$ is given in [10], where a scetch of its proof is also presented. There are r parameters $x_s^{(i_s)}$, $1\leq i_s\leq\mu_s$, $s\in\bar{\mathcal{D}}$ in $\underset{\mathcal{D}_{\mu,\theta}}{\nabla} f$. Denote $S\overset{def}{=}I^r$. Notation $\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{Y(K\times S)}$ means, that norm is taken through all x_i, $i=\overline{1,n}$, $x_s^{(i_s)}$, $1\leq i_s\leq\mu_s$, $s\in\bar{\mathcal{D}}$, i.e. $\underset{\mathcal{D}_{\mu,\theta}}{\nabla} f(x)$ is considered to be an element of space $Y(K\times S)$.

Now define power of the best approximation $E(f,\mathcal{D},\mu)_{Y(K)}$.

T h e o r e m 2. I. For any $f\in C(K)$

(3) $$\mathcal{A}(\mu_{\mathcal{D}})\|\underset{\mathcal{D}_{\mu,\theta}}{\nabla} f\|_{C(K\times S)}\leq E(f,\mathcal{D},\mu)_{C(K)}\leq B(\mu_{\mathcal{D}})\|\underset{\mathcal{D}_{\mu,\theta}}{\nabla} f\|_{C(K\times S)};$$

II. For any $f\in L_p(K)$, $0<p\leq\infty$

(4) $$C(\mu_{\mathcal{D}},P)\|\nabla_{\mathcal{D}_{\mu,0}} f\|_{L_p(K\times S)} \le E(f,\mathcal{D},\mu)_{L_p(K)} \le D(\mu_{\mathcal{D}},P)\|\nabla_{\mathcal{D}_{\mu,0}} f\|_{L_p(K\times S)},$$

where

$$B(\mu_{\mathcal{D}})=\left(\|(\Pi x)_{\mathcal{D}}\|_{C(S)}\right)^{-1};\ A(\mu_{\mathcal{D}})=\Big(\sum_{\eta\subset\mathcal{D}}\prod_{s\in\bar\eta}\mu_s\Big)^{-1}\cdot B(\mu_{\mathcal{D}});$$

$$D(\mu_{\mathcal{D}},P)=\left(\|(\Pi x)_{\mathcal{D}}\|_{L_p(S)}\right)^{-1};$$

$$C(\mu_{\mathcal{D}},P)=\Big(\sum_{\eta\subset\mathcal{D}}\prod_{s\in\bar\eta}\mu_s\Big)^{-\gamma}\cdot D(\mu_{\mathcal{D}},P);\ \gamma=\max(1,1/p).$$

(Note that the statements of theorem 2 for functions $f\in C(K)$ was formulated in [10])

P r o o f of theorem 2. Take a collection of fixed $x_s^{(i_s)}\in I$, $1\le i_s\le\mu_s$, $s\in\bar{\mathcal{D}}$.

Denote $\nabla^{|\eta|}$ summands in (2) corresponding to each collection containing $|\eta|$ elements, i.e.

$$\nabla_{\mathcal{D}_{\mu,0}} f=\sum_{|\eta|=0}^{|\mathcal{D}|}\nabla^{|\eta|} f.$$

Then

(5) $$\nabla_{\mathcal{D}_{\mu,0}} f=\nabla^{|\emptyset|} f+\nabla^{|1|} f+\sum_{|\eta|=2}^{|\mathcal{D}|}\nabla^{|\eta|} f.$$

By definition $\nabla^{|\emptyset|} f=f(x)(\Pi x)_{\mathcal{D}}$.

Further,

$$\nabla^{(1)} f=\sum_{\delta\in\mathcal{D}}\sum_{k_\delta=1}^{\mu_\delta}(-1)^{|k_\delta|-|\delta|+1} f(\overset{\vee}{x}{}_\delta^{(k_\delta)})(\Pi x)_{\delta}^{\ne k}(\Pi x)_{\mathcal{D}\setminus\delta}.$$

As far as $\forall\eta\subset\mathcal{D}$

$$(\Pi x)_{\eta}^{\ne k}=\prod_{s\in\bar\eta}\prod_{\substack{j_s=1\\ j_s\ne k_s}}^{\mu_s}(x_s-x_s^{(j_s)})\cdot\prod_{\substack{1\le i_s<j_s\le\mu_s\\ i_s,j_s\ne k_s}}(x_s^{(i_s)}-x_s^{(j_s)}),$$

is polynomial on variables x_s, of power μ_s-1, $s\in\bar\eta$, then $\nabla^{(1)} f$ is a quasipolynomial of power $\mu_{\mathcal{D}}-1$ defined by collection (5). Besides under condition $|\eta|\geq 2$ for each η expression

$$\sum_{k_\eta=1}^{\mu_\eta} (-1)^{|k_\eta|-|\bar\eta|+|\eta|} f(\check{x}_\eta^{(k_\eta)}) \left(\prod_{\neq k} x\right)_\eta \left(\prod x\right)_{\bar{\mathcal{D}}\setminus\bar\eta}$$

is a special case of some $P_{\mu_\delta}(x)$, $\delta\in\mathcal{D}$.
Therefore

$$\nabla^{(1)} f + \sum_{|\eta|=2}^{\mathcal{D}} \nabla^{|\eta|} f = \prod\nolimits_{\mu}^{*\mathcal{D}}(x) ,$$

i.e. is a specific quasipolynomial of power $\mu_{\mathcal{D}}-1$ defined by collection (5).

Consequently

$$\nabla_{\mathcal{D}_{\mu,\theta}} f = f(x)\left(\prod x\right)_{\mathcal{D}} + \prod\nolimits_{\mu}^{*\mathcal{D}}(x) .$$

Therefore

$$\left\|\nabla_{\mathcal{D}_{\mu,\theta}} f\right\|_{C(K)} = \left\| f(x)\cdot\left(\prod x\right)_{\mathcal{D}} + \prod\nolimits_{\mu}^{*\mathcal{D}}(x)\right\|_{C(K)} \leq \left\|\nabla_{\mathcal{D}_{\mu,\theta}} f\right\|_{C(K\times S)} \Longrightarrow$$

$$E\left[f(x)\cdot\left(\prod x\right)_{\mathcal{D}}, \mathcal{D}, \mu\right]_{C(K)} \leq \left\|\nabla_{\mathcal{D}_{\mu,\theta}} f\right\|_{C(K\times S)} .$$

But

$$E\left[f(x)\left(\prod x\right)_{\mathcal{D}}, \mathcal{D}, \mu\right]_{C(K)} = \left|\left(\prod x\right)_{\mathcal{D}}\right|\cdot E(f,\mathcal{D},\mu)_{C(K)} .$$

Consequently

$$\left\|\left(\prod x\right)_{\mathcal{D}}\right\|_{C(S)}\cdot E(f,\mathcal{D},\mu)_{C(K)} \leq \left\|\nabla_{\mathcal{D}_{\mu,\theta}} f\right\|_{C(K\times S)} \Longrightarrow$$

$$E(f,\mathcal{D},\mu)_{C(K)} \leq B(\mu_{\mathcal{D}})\left\|\nabla_{\mathcal{D}_{\mu,\theta}} f\right\|_{C(K\times S)} . \tag{6}$$

Let us establish lower bound in (3). As long as operator $\nabla_{\mathcal{D}_{\mu,\theta}} f$ is linear, for an arbitrary quasipolynomial $\prod_{\mu}^{\mathcal{D}}(x)$, by theorem I we have

$$\left|\nabla_{\mathcal{D}_{\mu,\theta}} f\right| = \left|\nabla_{\mathcal{D}_{\mu,\theta}}\left(f - \prod\nolimits_{\mu}^{\mathcal{D}}\right)\right| \leq \left(1 + \sum_{\delta\in\mathcal{D}}\prod_{s\in\delta}\mu_s + \right.$$

$$+\sum_{\delta_i,\delta_j\in\mathcal{D}}\prod_{s\in\delta_i\cup\delta_j}\mu_s+\dots+\prod_{s\in\bigcup_{i=1}^{|\mathcal{D}|}\delta_i}\mu_s)\cdot\|f-\Pi^{\mathcal{D}}_{\mu}\|_{C(K)}\cdot\|(\Pi x)_{\mathcal{D}}\|_{C(S)}=$$

$$=(\sum_{\eta\subset\mathcal{D}}\prod_{s\in\eta}\mu_s)\cdot\|f-\Pi^{\mathcal{D}}_{\mu}\|_{C(K)}\cdot\|(\Pi x)_{\mathcal{D}}\|_{C(S)}\Longrightarrow$$

$$\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{C(S)}(\sum_{\eta\subset\mathcal{D}}\prod_{s\in\eta}\mu_s)^{-1}\cdot(\|(\Pi x)_{\mathcal{D}}\|_{C(S)})^{-1}\le\|f-\Pi^{\mathcal{D}}_{\mu}\|_{C(K)}\Longrightarrow$$

(7) $$A(\mu_{\mathcal{D}})\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{C(K\times S)}\le E(f,\mathcal{D},\mu)_{C(K)}\,.$$

In case of space L_p we have

$$\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{L_p(K)}=\|f(x)(\Pi x)_{\mathcal{D}}-\Pi^{*\mathcal{D}}_{\mu}(x)\|_{L_p(K)}\ge\inf_{\Pi^{\mathcal{D}}_{\mu}(x)\in L_p(K)}\|f(x)(\Pi x)_{\mathcal{D}}-\Pi^{\mathcal{D}}_{\mu}(x)\|_{L_p(K)}=$$

$$=|(\Pi x)_{\mathcal{D}}|\cdot E(f,\mathcal{D},\mu)_{L_p(K)}\Longrightarrow$$

$$E(f,\mathcal{D},\mu)_{L_p(K)}\cdot\|(\Pi x)_{\mathcal{D}}\|_{L_p(S)}\le\|\,\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{L_p(K)}\|_{L_p(S)}\Longrightarrow$$

(8) $$E(f,\mathcal{D},\mu)_{L_p(K)}\le\mathcal{D}(\mu_{\mathcal{D}},p)\cdot\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{L_p(K\times S)}\,.$$

We shall establish lower bound in (4) separately for $p\ge 1$ and for $0<p<1$.

Under $p\ge 0$ for $\forall\,\Pi^{\mathcal{D}}_{\mu}(x)\in L_p(K)$

(9) $$\|\nabla_{\mathcal{D}_{\mu,\theta}}f\|_{L_p(K\times S)}=\|\nabla_{\mathcal{D}_{\mu,\theta}}(f-\Pi^{\mathcal{D}}_{\mu})\|_{L_p(K\times S)}=$$

$$\|\sum_{|\eta|=0}^{|\mathcal{D}|}\sum_{\eta\subset\mathcal{D}}\sum_{k_\eta=1}^{\mu_\eta}(-1)^{|k_\eta|-|\eta|+|\eta|}[f-\Pi^{\mathcal{D}}_{\mu}](\overset{\vee}{x}{}^{(k_\eta)}_{\eta})(\prod_{\ne k}x)_{\eta}(\Pi x)_{\mathcal{D}\setminus\eta}\|_{L_p(K\times S)}\le$$

$$\le \left\| \left(f - \Pi_{\mu}^{\mathcal{D}}\right) (\Pi x)_{\mathcal{D}} \right\|_{L_p(K\times S)} +$$

$$+ \sum_{|\eta|=1}^{|\mathcal{D}|} \sum_{\eta\subset\mathcal{D}} \sum_{K_\eta=1}^{\mu_\eta} \left\| \left[\left(f - \Pi_{\mu}^{\mathcal{D}}\right)\right] \left(\overset{\vee}{X}_\eta^{(K_\eta)}\right) \left(\prod_{\neq K} x\right)_\eta \cdot (\Pi x)_{\bar{\mathcal{D}}\setminus\bar{\eta}} \right\|_{L_p(K\times S)} \le$$

$$\le \left\| f(x) - \Pi_{\mu}^{\mathcal{D}}(x) \right\|_{L_p(K)} \cdot \left\| (\Pi x)_{\mathcal{D}} \right\|_{L_p(S)} +$$

$$+ \sum_{|\eta|=1}^{|\mathcal{D}|} \sum_{\eta\subset\mathcal{D}} \sum_{K_\eta=1}^{\mu_\eta} \left\| \left(f - \Pi_{\mu}^{\mathcal{D}}\right)\left(\overset{\vee}{X}_\eta^{(K_\eta)}\right) \right\|_{L_p(K)} \cdot \left\| \left(\prod_{\neq K} x\right)_\eta \cdot (\Pi x)_{\bar{\mathcal{D}}\setminus\bar{\eta}} \right\|_{L_p(S)} .$$

As it was mentioned above, norm $\|\cdot\|_{L_p(K\times S)}$ is taken with respect to variables $x_i^{(0)} \overset{def}{=} x_i$, $i=\overline{1,n}$ and $x_s^{(i_s)}$, $1\le i_s\le \mu_s$, $s\in\bar{\mathcal{D}}$. Here the n norm $\|[f-\Pi_{\mu}^{\mathcal{D}}](\overset{\vee}{x}_\eta^{(K_\eta)})\|_{L_p(K)}$ is taken in respect to variables $\hat{X}_\eta$ and $x_\eta^{(K_\eta)}$, and the norm $\|(\prod_{\neq K} x)_\eta\, (\Pi x)_{\bar{\mathcal{D}}\setminus\bar{\eta}}\|_{L_p(S)}$ is taken in respect to remaining variables.

Further we have from (9)

$$\left\| \nabla_{\mathcal{D}_{\mu,\theta}} f \right\|_{L_p(K\times S)} \le \left\| \left[f - \Pi_{\mu}^{\mathcal{D}}\right](x) \right\|_{L_p(K)} \left\| (\Pi x)_{\mathcal{D}} \right\|_{L_p(S)} \cdot \left(\sum_{\eta\subset\mathcal{D}} \prod_{s\in\bar{\eta}} \mu_s \right) \Longrightarrow$$

$$\left(\sum_{\eta\subset\mathcal{D}} \prod_{s\in\bar{\eta}} \mu_s \right)^{-1} \left\| (\Pi x)_{\mathcal{D}} \right\|_{L_p(S)}^{-1} \left\| \nabla_{\mathcal{D}_{\mu,\theta}} f \right\|_{L_p(K\times S)} \le \left\| f - \Pi_{\mu}^{\mathcal{D}} \right\|_{L_p(K)}$$

or

$$C(\mu_{\mathcal{D}}, p) \left\| \nabla_{\mathcal{D}_{\mu,\theta}} f \right\|_{L_p(K\times S)} \le E(f,\mathcal{D},\mu)_{L_p(K)} . \tag{10}$$

At last, for $0<p<1$,

$$\left\| \nabla_{\mathcal{D}_{\mu,\theta}} f \right\|_{L_p(K\times S)}^p \le \left\| \left[f - \Pi_{\mu}^{\mathcal{D}}\right](x) \right\|_{L_p(K)}^p \cdot \left\| (\Pi x)_{\mathcal{D}} \right\|_{L_p(S)}^p \cdot \left(\sum_{\eta\subset\mathcal{D}} \prod_{s\in\bar{\eta}} \mu_s \right) \Longrightarrow$$

$$\left(\sum_{\eta\in\mathcal{D}} \prod_{s\in\bar{\eta}} \mu_s \right)^{-1} \left\| \nabla_{\mathcal{D}_{\mu,\theta}} f \right\|_{L_p(K\times S)}^p \left\| (\Pi x)_{\mathcal{D}} \right\|_{L_p(S)}^{-p} \le E(f,\mathcal{D},\mu)_{L_p(K)}^p \Longrightarrow$$

$$(11)\qquad \Big(\sum_{\eta\in\mathcal{D}}\prod_{s\in\bar\eta}\mu_s\Big)^{-\frac{1}{p}}\|(\Pi x)_{\mathcal{D}}\|^{-1}_{L_p(S)}\cdot\|\nabla_{\mathcal{D}_{\mu,0}}f\|_{L_p(K\times S)}\le E(f,\mathcal{D},\mu)_{L_p(K)}.$$

Relations (6)-(7) prove part I, and (8),(10)-(11) part II of theorem 2.

R e m a r k. Theorem 2 is valid also, when variables $x_s\in I$ are arbitrary and points are equidistant

$$x_s^{(i_s)}:\ x_s^{(i_s+1)}=x_s^{(i_s)}+h_s,\ h_s>0,\ 1\le i_s\le\mu_s-1,\ s\in\bar{\mathcal{D}}.$$

One can convince himself in this by following the proof of this theorem.

3. P r e p o s i t i o n. Whitney's theorem follows from theorem 2.I.

P r o o f. Let $n=1$, $d(\cdot,\dots,\cdot)$ - is diameter of the set, consisting of points in brackets, $A(\mu)$, $A_1(\mu)$ - are constants depending only on μ.

$$(12)\qquad \begin{gathered} x,x_i\in[0,1],\ x_{i+1}=x_i+h,\ \forall i=\overline{1,\mu-1},\ x_\mu\in[0,1],\ h\ne0\\ [x,x_1,\dots,x_\mu;f,x]\overset{def}{=}[x,x_1,\dots,x_\mu]_f\prod_{i=1}^{\mu}(x-x_i),\end{gathered}$$

where $[x,x_1,\dots,x_\mu]$ - is separated difference of power μ of function of one variable $f=f(x)$. Let us use relations courteously presented to me by P.M.Tamrasov:

$$(13)\qquad \begin{gathered}[x,x_1,\dots,x_n;f,x]\le A(\mu)\Big[\frac{d(x,x_1,\dots,x_\mu)}{d(x_1,\dots,x_\mu)}\Big]^{\mu-1}\omega_{\mu,2}(f,d(x,x_1,\dots,x_\mu))\le\\ \le A_1(\mu)\Big[\frac{d(x,x_1,\dots,x_\mu)}{d(x_1,\dots,x_\mu)}\Big]^{\mu-1}\omega_{\mu,1}(f,d(x,x_1,\dots,x_\mu)),\end{gathered}$$

where $\omega_{n,N}(f,\delta)$ - N - is uniform modulus of smoothness, and also $\omega_{n,1}(f,\delta)$ -is usual(arithmetic) modulus of smoothness (see [7]) The first inequality in (13) is a special case of lemma 5.7.1 from [7] ,p.224, and the second is a special case of theorem 15 and its corollary from [8] ,p.18.

By virtue of the remark to theorem 2 in case $n=1$ and validity of conditions (12) the right inequality from (3) can be rewritten in the form

$$(14)\qquad E(f,P_{\mu-1})_{C(I)}\le B(\mu)\|\nabla^*_{\mathcal{D}_{\mu,0}}f\|,$$

where the norm at the right hand side is taken with respect to $h, x, x_1 \in I,\ x_1+(\mu-1)h \in I$ and

$$\nabla^*_{\mathcal{D}_{\mu,0}} f = f(x) \prod_{1\le i<j\le\mu} (x_i-x_j) + \sum_{k=1}^{\mu} (-1)^k f(x_k) \prod_{\substack{0\le i<j\le\mu \\ i,j\ne k}} (x_i-x_j) .$$

Here it is accepted $x_0 \stackrel{def}{=} x$. We have

$$\nabla^*_{\mathcal{D}_{\mu,0}} f = \Big(\sum_{k=0}^{\mu} \frac{f(x_k)}{\prod\limits_{\substack{j=0 \\ j\ne k}}^{\mu} (x_k-x_j)} \Big) \cdot \prod_{0\le i<j\le\mu} (x_i-x_j) =$$

(15)

$$= [x, x_1, \ldots, x_\mu] \prod_{0\le i<j\le\mu} (x_i-x_j) .$$

Using (13) and (15) in (14) gives

$$E(f, P_{\mu-1})_{C(I)} \le B(\mu) \Big\| [x,x_1,\ldots,x_\mu] \cdot \prod_{j=1}^{\mu} (x-x_j) \cdot \prod_{1\le i<j\le\mu} (x_i-x_j) \Big\| \le$$

(16)

$$\le B(\mu) \Big\| A_1(\mu) \Big[\frac{d(x,x_1,\ldots,x_\mu)}{d(x_1,\ldots,x_\mu)} \Big]^{\mu-1} \omega_{\mu,1}\big(f, d(x,x_1,\ldots,x_\mu)\big) \cdot \prod_{1\le i<j\le\mu} (x_i-x_j) \Big\| .$$

Further $d(x_1,\ldots,x_\mu) = |x_\mu - x_1| = (\mu-1)h$,

$$d(x,x_1,\ldots,x_\mu) \le 1;\ \Big| \prod_{1\le i<j\le\mu} (x_i-x_j) \Big| = \Big| \prod_{j=2}^{\mu} (x_1-x_j) \Big| \cdot \Big| \prod_{2\le i<j\le\mu} (x_i-x_j) \Big| =$$

$$= h \cdot 2h \ldots (\mu-1)h \cdot \Big| \prod_{2\le i<j\le\mu} (x_i-x_j) \Big| \le h^{\mu-1} \cdot (\mu-1) .$$

Then the right hand side of inequality (16) will have the form

$$B(\mu) \cdot \Big\| A_1(\mu) \cdot \frac{1}{(\mu-1)^{\mu-1} h^{\mu-1}} \cdot h^{\mu-1} \cdot (\mu-1)!\, \omega_\mu(f,1) \Big\| ,$$

i.e.

$$E(f, P_\mu)_{C(I)} \le C_\mu \sup_{0\le x,\, x+(\mu-1)h\le 1} \big| \Delta_h^{\mu} f \big| ,$$

which is Whitney's theorem.

4. Exact annihilator of class of quasipolynomials $(\Pi^{\mathcal{D}}_{\mu})_Y$ can be expressed, through separated differences. We shall need notations

$$b_s \overset{def}{=} \prod_{j_s=1}^{\mu_s} (x_s - x_s^{(j_s)}) \; ; \; a_s = b_s^{-1} \; ; \; \forall \alpha \subset \bar{D}, \; a_\alpha \overset{def}{=} \prod_{s \in \alpha} a_s .$$

If $\alpha = \{ s_1, \ldots s_k \}$, then $a_\alpha = a_{s_1} \ldots a_{s_k} \overset{def}{=} a_{s_1 \ldots s_k}$.
It is known that the separated difference of function f with respect to variable x_s of power μ_s can be defined by formula

$$[x_s] f \overset{def}{=} [x_s^{(0)} \ldots x_s^{(\mu_s)}] f = \sum_{k_s=0}^{\mu_s} \frac{f(\overset{\vee}{x}_s^{(k_s)})}{\prod_{\substack{i_s=0 \\ i_s \neq k_s}}^{\mu_s} (x_s^{(k_s)} - x_s^{(i_s)})} .$$

Define in the following way the separated difference of function f with respect to group of variables x_η of power μ_η (with respect to x_i of power μ_i , $i \in \bar{\eta}$)

$$[x_\eta] f = [x_{\eta(1)}, \ldots, x_{\eta(|\bar{\eta}|)}] = \sum_{k_\eta=0}^{\mu_s} \frac{f(\overset{\vee}{x}_\eta^{(k_\eta)})}{\prod_{s \in \bar{\eta}} \prod_{\substack{j_s=0 \\ j_s \neq k_s}}^{\mu_s} (x_s^{(k_s)} - x_s^{(j_s)})} .$$

T h e o r e m. 3. Exact annihilator of class of quasipolynomials can be defined by using the separated differences by formula

$$\nabla_{D_{\mu,\theta}} f = \sum_{\eta \subset D} (-1)^{|\eta|} \sum_{\nu \subset \bar{\eta}} (-1)^{|\nu|} [x_\nu] f \cdot b_\nu \cdot \left(\prod x \right)_D .$$

5. Historical reference. Earlier for isolated special cases

estimations $E(f,\mathcal{D},\mu)_{Y(T)}$ were obtained, where finite differences served as annihilators. Mainly the case of $n=1$ (approximation of the function of a single variale by polynomials) was investigated. The first result here belongs to Whitney [1-2] (the case, when T is a closed interval from R). A new proof of this result is given in [6] . Further cases $T=I$, $Y=L_p$, $p \geq 1$ [3] and $Y=L_p$, $0<p<1$ [5] where treated. For an arbitrary n the case $\delta_i=\{i\}$, $i=\overline{1,n}$, $T=K$, $Y=L_p$, $1<p\leq\infty$ was considered in [4] , and the case $\forall\mathcal{D}$, $\mu_i=1$, $i=\overline{1,n}$ (approximation by sums of functions of fewer variables), where $Y=C(T)$ or $L_p(T)$, $0<p\leq\infty$; $T=K$, $\mathbb{T}^n$ (n - dimensional torus) or R in [9] . The last case is valid both for real and complex valued approximated function f of real variables.

Finally, [10] presents scheme of the proof of the statement that, the union of operators $\nabla_{\mathcal{D},\mu,\theta} f$ is an exact annihilator of class of quasipolynomials $\Pi_{\mu}^{\mathcal{D}}(x)\in C(K)$ and formulates theorem 2.I about the degree of the best approximation .

As it was shown above in case $n=1$ Whitney's theorem follows from theorem 2.I.

Several special cases of quasipolynomials for the first time were presented by Ya.S.Bugrov [11], H.S.Shapiro [12] and M.K.Potapov (for example [13]).

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