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ON APPROXIMATION BY SUPERPOSITIONS IN THE MIXED NORM

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1. Let real function $\Phi = \Phi(x)$ $x=(x_1, x_2, \dots, x_n)$ be defined on bounded set Q from n-dimensional euqleadean space $R_n = R(x_1, \dots, x_n)$ is given by inequalities

$$Q = \begin{cases} \alpha_i(x_{\overline{i+1,n}}) \le x_i \le \beta_i(x_{\overline{i+1,n}}), & i=\overline{1,n-1} \\ \alpha_n = a \le x_n \le b = \beta_n, & \end{cases}$$

where $\alpha_i(x_{\overline{i+1,n}}), \beta_i(x_{\overline{i+1,n}})$ fixed real functions,

$$\|\Phi\|_{p_i} \overset{def}{=} \left(\int_{\alpha_i(x_{\overline{i+1,n}})}^{\beta_i(x_{\overline{i+1,n}})} |\Phi(x)|^{p_i} dx_i \right)^{\frac{1}{p_i}}, \quad p_i > 0, \quad i=\overline{1,n}.$$

Denote through $L_{\bar p}(Q)$ $\bar p = (p_1, p_2, \dots, p_n)$ space of functions, $\Phi=\Phi(x)$ for which the following integral exists and is finite

$$\|\Phi\|_{\bar p} \overset{def}{=} \left\| \left(\| \dots \|\Phi\|_{p_1} \dots \|_{p_{n-1}} \right) \right\|_{p_n}.$$

We shall call function $\Phi=\Phi(x)$ of many variables increasing if it is increasing function of each variable, i.e. if $x' \le x''$ ($x_i' \le x_i''$ $i=\overline{1,n}$) then $\Phi(x') \le \Phi(x'')$.
Take group of variables $t_\nu \subset \{x_1, x_2, \dots, x_n\}$ $t_\nu \not\subset t_\mu$, $\nu \ne \mu$, $\nu, \mu = \overline{1,m}$. Denote as $\sum$ a class of functions of the form

$$\sum \varphi_\nu \overset{def}{=} \sum_{\nu=1}^{m} \varphi_\nu(t_\nu) \in L_{\bar p}(Q),$$

where functions $\varphi_\nu(t_\nu)$ are defined on Q_ν -projections of set Q on to space $R(t_\nu)$. We'll call value

$$E_{\bar p}(\Phi, \Sigma) = \inf_{\sum \varphi_\nu \in \Sigma} \|\Phi - \sum \varphi_\nu\|_{\bar p}$$

the best approximation of function Φ by the set of sums of functions of fewer variables $\sum$ in space with mixed norm $\mathcal{L}_{\bar{p}}(Q)$.
Let $r_i \in N$ $i \in \{1,2,\dots,n\} \stackrel{def}{=} \bar{n}$, Q_i - projection of Q onto $R(x_i)$, $\tilde{Q} = Q_1 \times \dots \times Q_n$. Take a partioning of each Q_i into r_i parts

$$x_i^0 = \min_{x_i \in Q_i} x_i < x_i^{(1)} < \dots < x_i^{(r_i)} = \max_{x_i \in Q_i} x_i, \quad 1 = \overline{1,n}.$$

Consider expression

$$V(f) = \sum_{i \in \bar{n}} \sum_{k_i=0}^{r_i-1} \left| f(x^{(k_i+1)}) - f(x^{(k_i)}) \right|.$$

If

$$\sup V(f) \stackrel{def}{=} \underset{\tilde{Q}}{V}(f) < +\infty,$$

where $\sup$ is taken among all feasible partionings, then we shall call f function with bounded variation on $\tilde{Q}$. If $\underset{\tilde{Q}}{V}(f) \le M$ we shall write $f \in V_M$.

Denote through $\sum_{V_M}$ class of functions belonging to $\sum$, variations of which are bounded by number M.

T h e o r e m 1. There is the best approximating element in class $\sum_{V_M}$ for each function $\Phi \in \mathcal{L}_{\bar{p}}(Q)$.

The following auxiliary propositions are used to prove theoremI.

L e m m a I. Let $F = \{f(x)\}, x \in Q \subset R_n$ is infinite family of increasing functions defined on $\tilde{Q}$. If all functions of the family are bounded by the same number $|f(x)| \le K$ then a subsequence $\{f_n(x)\}$ can be extracted from F, which converges to some increasing function $\varphi(x)$ at every point of $\tilde{Q}$.

L e m m a 2. (Helly's theorem). Let infinite family of functions $F = \{f(x)\}$ is given on set $\tilde{Q}$. If all functions of the family and their total variations are bounded by the same number

$$|f(x)| \le M, \quad \underset{\tilde{Q}}{V}(f) \le M,$$

then a subsequence $\{f_n(x)\}$ can be extracted from family F, which converges to some function $\varphi(x)$ at every point of $\tilde{Q}$ and has the bounded variation.

Let $\alpha \subset \{1,2,\dots,n\}$, $0 < p_i < 1$, if $i \in \alpha$, $p_i \ge 1$, if $i \in \bar{n} \setminus \alpha$,

$$p^* = \begin{cases} \min P_i, \quad i \in \alpha & , \text{ if } \alpha \neq \emptyset \\ 1, & , \text{ if } \alpha = \emptyset . \end{cases}$$

L e m m a 3. For arbitrary $f, g \in \mathcal{L}_{\bar{p}}(Q)$

$$\| f - g \|_{\bar{p}}^{p^*} \leq \| f \|_{\bar{p}}^{p^*} + \| g \|_{\bar{p}}^{p^*} .$$

L e m m a 4. [I] Let a uniformly bounded family of functions $\left\{ \sum_{\nu=1}^{m} \varphi_{\nu,\mu}(t_\nu) \right\}$ $\mu = 1, 2, \ldots$ is defined on parallelepiped $\tilde{Q} = [Q_1, \ldots, Q_n]$. Then families $\{\varphi^*_{\nu,\mu}(t_\nu)\}$ $\nu = \overline{1,m}$ exist which are uniformly bounded on Q_ν and for which

$$\sum_{\nu=1}^{m} \varphi^*_{\nu,\mu}(t_\nu) = \sum_{\nu=1}^{m} \varphi_{\nu,\mu}(t_\nu)$$

on $\tilde{Q}$.

R e m a r k. It can be followed from the way of constrnction of families $\left\{ \varphi^*_{\nu,\mu}(t_\nu) \right\}$ when prooving lemma 4, that

$$\forall \mu = 1, 2, \ldots \quad \sum_{\nu=1}^{m} \varphi_{\nu,\mu}(t_\nu) \in V_M \Longrightarrow \varphi^*_{\nu,\mu}(t_\nu) \in V_M .$$

2. Fixe natural numbers m and K. Consider families of functions $\{\varphi_{ij}(\tau_{ij})\}$ and $\{\psi_{ij}(\tau_{ij})\}$, $\tau_{ij} \subset \{x_1, \ldots, x_m\}$, $i = \overline{1,m}$, $j = \overline{1,K}$. Obviously $t \overset{def}{=} \bigcup_{i,j} \tau_{ij} \subset \{ x_1, \ldots, x_n \}$. Associate with each pair of such collection expression

$$(1) \quad \sigma(t) \overset{def}{=} \sum_{(i,j) \in \eta} \varphi_{ij}(\tau_{ij}) + \sum_{i=1}^{m} \prod_{j=1}^{K} \psi_{ij}(\tau_{ij}), \quad \eta = \left\{ (i,j) \middle| \tau_{ij} \not\subset \tau_{i'j'}, (i,j) \neq (i',j') \right\}$$

Denote G the class of all functions $\sigma = \sigma(t) \in \mathcal{L}_{\bar{p}}(Q)$ of the form (1). Consider the best approximation

$$E_{\bar{p}}(\Phi, G) = \inf_{\sigma \in G} \| \Phi(x) - \sigma(t) \|_{\bar{p}} .$$

Let τ_α is a subset of some τ_{ij}. Denote through $Q(\tau_\alpha)$ the totality of points $x \in Q$, with fixed coordinates $x \setminus \tau_\alpha$. Function $\Phi_{Q(\tau_\alpha)}(\tau_\alpha)$ related to subset $Q(\tau_\alpha) \subset Q$, defined on subset $Q(\tau_\alpha)$ and which coincides on it with function on $\Phi(x)$

$$\Phi_{Q(\tau_\alpha)}(\tau_\alpha) = \Phi(x), \; x \in Q(\tau_\alpha)$$

will be called subfunction of $\Phi(x)$.

Denote through Q_α the projection of set $Q(\tau_\alpha)$ on to space $R(\tau_\alpha)$, and through $\Phi_{Q_\alpha}(\tau_\alpha)$ the function, defined on Q_α in the following way

$$\Phi_{Q_\alpha}(\tau_\alpha) = \Phi_{Q_\alpha}(x)$$

Denote through $B_Q^{(\alpha)}$ the projection of set Q on to space $R(x \setminus \tau_\alpha)$. Associate with each point from $B_Q^{(\alpha)}$ a function $\Phi_{Q_\alpha}(\tau_\alpha)$. Then a family $\{\Phi_{Q_\alpha}(\tau_\alpha)\}$ which will be called α - family of function $\Phi(x)$, will be associated with set $B_Q^{(\alpha)}$.

D e f i n i t i o n. Family $\{R^\sigma_{Q_\alpha}(\tau_\alpha)\}$ will be called equivalent to family $\{\Phi_{Q_\alpha}(\tau_\alpha)\}$ if the first is α - family of function of the form

$$R^\sigma(x) = \Phi(x) - \sigma(t).$$

Let $x \setminus \tau_\alpha \overset{def}{=} \bar{\tau}_\alpha$. Then $R^\sigma(x) = R^\sigma(\tau_\alpha, \bar{\tau}_\alpha)$. Fixe $\bar{\tau}_\alpha = \bar{y}_\alpha$.
We shall call expression

$$\Delta^{(\alpha)}(R^\sigma) = \inf_{\bar{y}_\alpha} \left\| R^\sigma(\tau_\alpha, \bar{\tau}_\alpha) - R^\sigma(\tau_\alpha, \bar{y}_\alpha) \right\|_{\bar{P}}$$

the main diameter of α - family of function $R^\sigma(x) = \Phi(x) - \sigma(t)$.
The main veritable diameter of α - family of function $\Phi(x)$ is defined by

$$\Delta_0^{(\alpha)}(\Phi) = \inf \Delta^{(\alpha)}[R^\sigma],$$

where $\inf$ is taken among families, which are equivalent with α - family of function $\Phi(x)$.

T h e o r e m 2. The following equality is valid

$$\Delta_0^{(\alpha)}(\Phi) = E_{\bar{p}}(\Phi, G) .$$

Earlier the theorem of existence in $L_{\bar{p}}(Q)$ $p_i \geq 1$, $i=\overline{1,n}$ for approximating by sums of functions of a single variable was established in [2] , and in space C for approximating by sums of functions of fewer variables in [1] .

R E F E R E N C E S

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