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GIVEN TWO SPACES OF GENERALIZED DIRICHLET POLYNOMIALS, WHICH ONE IS CLOSER TO x^c ?

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1. Notation. The letters s, s' are used to denote finite increasing sequences of real numbers; if $s = (s_1, s_2, \dots, s_\ell)$, $s_1 \le s_2 \le \dots \le s_\ell$, the distinct values of s_j, $j = 1,2,\dots,\ell$ are denoted $c_1, c_2, \dots, c_k$; $c_1 < c_2 < \dots < c_k$; the multiplicity with which c_j appears in s is denoted m_j. The vector-space of generalized Dirichlet polynomials generated by

$$x^{c_j}(\log x)^{d_q}, \quad q = 0,2,\dots,m_j - 1 \ ; \ j = 1,2,\dots,k$$

is denoted $V(s)$. If I is an interval on the real axis, and χ_I is its characteristic function, then

$$s(I) = \sum_j m_j \chi_I(c_j) ,$$

i.e. $s(I)$ is the number of s_j's in I, counting multiplicities.

The norm sign denotes any L_p-norm, $1 \le p \le +\infty$ on $[a,b]$, $0 \le a < b \le +\infty$; if $p = \infty$, it is assumed, unless otherwise stated, that $a > 0$. Finally,

$$(1) \qquad d(x^c, V(s)) = \operatorname{Min}\{\|x^c - y\| \,|\, y \in V(s)\} .$$

This expression is defined provided $x^c, x^{c_j} \in L_p(a,b)$ for $j = 1,2,\dots,k$; so, if $a = 0$, it will be assumed that c and c_j's are $> -\frac{1}{p}$, and, if $b = +\infty$, that they are $< -\frac{1}{p}$.

By the change of variable $x = t^{-1}$, resp. $x = t^\gamma$, $\gamma > 0$ or $x = e^t$, we can assume that $c \geq -\frac{1}{p}$ and that $h < +\infty$; we can reduce the two parameters c and h in the corollary below to one parameter, assuming, for example, that $h = 1$ (or $c = 1$) ; and we can write the elements of $V(s)$ as generalized exponential polynomials.

2. Main result. Let k be a non-negative integer. How should one choose integers $n_1, n_2, \ldots, n_\ell$; $0 \leq n_1 < n_2 < \ldots < n_\ell$, $n_j \neq k$, in order that the expression

$$\min_{A_j} \| x^k - \sum_j A_j x^{n_j} \| \tag{2}$$

becomes minimum?

A conjecture of G. G. Lorentz [7] was proved and extended [4, 11, see also 10] to give the following result:.

The function of $\bar{n} = (n_1, n_2, \ldots, n_\ell)$, defined by (2), attains its minimum for some $\bar{n}$ (which in general is not unique), and for each $\bar{n}^0 = (n_1^0, n_2^0, \ldots, n_\ell^0)$ at which the minimum is attained,

(3) the set $\{n_1^0, n_2^0, \ldots, n_\ell^0\} \cup \{k\}$ is a set of consecutive integers.

This follows immediately from "the improvement theorem for Cartesian systems" of P.W. Smith [11]. We present here a refinement of that theorem:

THEOREM. If $s'(I) \geq s(I)$ for every interval I containing c, then

$$d(x^c, V(s')) < d(x^c, V(s)),$$

(except in the two trivial cases: $s' = s$ or $c = s_j$ for some j).

Only minor modifications are necessary in the proof given in [11] in order to prove this theorem, the main new ingredient is the following elementary lemma.

LEMMA. Let

$$F(X) = \sum_{j=1}^{k} P_j(\log x) x^{c_j}, \tag{4}$$

where P_j is a polynomial of degree $m_j - 1$, $\sum_{j=1}^{k} m_j = \ell + 1$, have ℓ zeroes on $(0,+\infty)$. Let A_j be the coefficient by the highest power of $\log x$ in $P_j(\log x)$. Then the sign of F for x sufficiently large is $(-1)^{v_j}$ sign A_j where

$$v_j = \sum_{i=j+1}^{k} m_i \ , \ j = 1, 2, \ldots, k .$$

<u>Proof of lemma.</u> By replacing $\log x$ in (4) with $\frac{x^{\varepsilon} - 1}{\varepsilon}$, $\varepsilon > 0$, an ordinary Dirichlet polynomial Q_ε is obtained. If ε is sufficiently small, Q_ε has also ℓ zeroes and has length $\ell + 1$, and so-- by Descartes rule of signs--has coefficients alternating in sign. If ε is sufficiently small, then $c_j + \varepsilon(m_j - 1) < c_{j+1}$ for all j , the coefficient by $x^{c_j+\varepsilon(m_j-1)}$ is A_j , and the number of terms following $x^{c_j+\varepsilon(m_j-1)}$ is v_j , so that $\mathrm{sign}(A_k) = (-1)^{v_j}$ sign (A_j) . It is clear from (4) that, for large x, $F(x)$ has the sign of A_k .

From the Theorem one obtains the following analogue to (3):

<u>Corollary.</u> The function $d(x^c, V(s))$ attains its minimum on $\{s| \ |s_j - c| \geq h > 0 \ , \ j = 1,2,\ldots,\ell\}$ and if s^0 is a point at which the minimum is attained, then

(5) $$s_j^0 = c \pm h \text{ for every } j .$$

Estimates from below for (2), obtained in [5, 8 and 9], have been proved using (3); on the basis of this corollary similar estimates from below can be deduced for (1).

Estimates for coefficients of polynomials in [1, 2 and 3] provide also estimates from below for (2); they do not depend on (3) and one can deduce similar estimates from below for (1), using the continuity of $d(x^c, V(s))$ as a function of s . (The continuity of $d(x^c, V(s))$ follows from a compactness result - see Theorem 1 in [6]).

3. Observations and Open Questions. If the norm is the L_2-norm on $[0,1]$, then

$$d(x^c, V(s)) = \frac{1}{\sqrt{2s+1}} \prod_{j=1}^{\ell} \frac{|c - s_j|}{c + s_j + 1} ;$$

this formula is well known if $s_1 < s_2 < \dots < s_\ell$; by continuity it holds also for $s_1 \leq s_2 \leq \dots \leq s_\ell$. Using this formula one can compute $n_1^0, n_2^0, \dots, n_\ell^0$ in (3) for each k and ℓ, and $s_1^0, s_2^0, \dots, s_\ell^0$ in (5) for each c, h and ℓ, $c - h > -\frac{1}{2}$. Let ν denote the number of terms n_j^0 which are $> c$, and σ the number of terms s_j^0 which are $> c$. It turns out that ν takes different values between $\frac{\ell}{2}$ and ℓ when k and ℓ vary; but - when c, h and ℓ vary - σ takes always the same value ℓ. Thus, in the case of the $L_2(0,1)$ norm, the conclusion (5) of the Corollary can be replaced by

$$s_j^0 = c + h \quad \text{for all } j . \tag{6}$$

A similar asymmetry appears in the inequality

$$\min_{A} \max_{x \in [t,1]} |x^c - Ax^{c+1}| < \min_{B} \max_{x \in [t,1]} |x^c - Bx^{c-1}| \tag{7}$$

where c is assumed to be > 1, and $0 \leq t < 1$. (An efficient way to prove (7) is, after computing both sides of (7), to write the left-hand side as $\sup_{1>u>t} f(u)$ and the right-hand side as $\sup_{1>u>t} g(u)$, where $f(u) = \frac{u^c(1-u)}{1+u^{c+1}}$, $g(u) = \frac{u^{c-1}(1-u)}{1+u^{c-1}}$, and to check that $f(u) < g(u)$ for $0 < u < 1$.)

These examples of the asymmetry raise the question:

If $c > -\frac{1}{p}$, $s_j > -\frac{1}{p}$, $s_j' > -\frac{1}{p}$ and s' is obtained from s by replacing some term s_j, $s_j < c$ by $s_j' = c + (c - s_j)$, is $d(x^c, V(s')) < d(x^c, V(s))$?

Here is a conjecture:

If $c > 1$, then

$$\min_{a_j, |c_j - c| \geq 1} \left\| x^c - \sum_{j=1}^{\ell} a_j x^{c_j} \right\|_{L_\infty(0,1)} = \min_{\substack{P \text{ polynomials} \\ \text{of degree } \ell}} \| x^c(1 - x\,P(\log x)) \|_{L_\infty(0,1)}$$

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