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LAGRANGE INTERPOLATION FOR THE COMPLETED LAGUERRE ABSCISSAS

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1. Introduction. The aim of this paper is to show that Lagrange interpolation based on the Laguerre roots and the zero generates a convergent approximation process on $[0,\infty)$ for a wide class of functions. Moreover the higher derivatives of the interpolatory polynomials converge to the derivatives of the original function.

2. Historical remarks. Interpolating polynomials of degree 2n-1 based on the roots of nth Laguerre polynomials and the zero were introduced first by E. Egerváry and P. Turán [2], as the "most economical" stable interpolalation on $[0,\infty)$. Convergence theorem was proved by J. Balázs and P. Turán [1] and later this process was investigated by I. Joó [5] , [6] and [7].

Lagrange interpolation for the Laguerre abscissas and its convergence were treated by G. Freud [3] and P. Névai [8], [9] and [10].

We mention that convergence problems of the Hermite-Fejér interpolation on Laguerre roots were considered by J. Szabados [11], [12] and B. Háy [4]. Furthermore the latter author gave estimates for that case when the Laguerre roots were completed with the zero.

3. Results. Let

$$L_n(x) = \frac{e^x}{n!}(x^n e^{-x})^{(n)}, \qquad n=1,2,\ldots,$$

be the Laguerre polynomials of degree n with the usual normalization $L_n(0)=1$ and with zeros $0<x_{1n}<x_{2n}<\ldots<x_{nn}$. These polynomials are orthogonal on $[0,\infty)$ with respect to the weight function e^{-x}.

Let f be a function defined on $[0,\infty)$ and let denote by $Q_n(f;x)$ its Lagrange polynomial of degree n based on the Laguerre roots and the zero

$$Q_n(f;x) = \sum_{k=1}^{n} f(x_{kn}) \frac{x}{x_{kn}} \ell_{kn}(x) + f(0) L_n(x), \qquad n=1,2,\ldots,$$

where

$$\ell_{kn}(x) = \frac{L_n(x)}{(x-x_{kn}) L_n'(x_{kn})}, \qquad k=1,2,\ldots,n.$$

The following convergence theorems and estimates can be proved for $Q_n(f;x)$:

THEOREM 1. Let $f \in \mathrm{Lip}\,\gamma$, $1/2<\gamma\leq 1$. Then

$$|f(x) - Q_n(f;x)| = O(1)\, x^{1/2} e^{x/2} n^{-\gamma/2+1/4},$$

for $0\leq x\leq x_{nn}$.

Note the important fact

$$x_{nn} = C_n(n+1), \qquad 1/4<C_n<4,$$

for the greatest zero of $L_n(x)$, which follows from Szegő [13]

(6.31.13) e.g.

THEOREM 2. Suppose that $f^{(r)}$ exists for some $r \geq 1$ and $f^{(r)} \in \mathrm{Lip}\,\gamma$, $0 < \gamma \leq 1$. Then

$$|f(x) - Q_n(f;x)| = O(1)\, x^{1/2}\, e^{x/2}\, n^{-(r+\gamma)/2+1/4},$$

for $0 \leq x \leq x_{nn}$.

COROLLARY. The convergence is uniform in every finite subinterval of $[0,\infty)$ under the assumptions of Theorems 1 or 2.

The derivatives $Q_n^{(i)}(f;x)$ converge to $f^{(i)}(x)$ on the interval $(0,\infty)$, if $1 \leq i \leq \left[\frac{r}{2}\right]$:

THEOREM 3. Suppose that $f^{(r)}$ exists for some $r \geq 1$ and $f^{(r)} \in \mathrm{Lip}\,\gamma$, $1/2 < \gamma \leq 1$. Then

$$|f^{(i)}(x) - Q_n^{(i)}(f;x)| = O(1)\, x^{-i}\, e^{x}\, n^{-(r+\gamma)/2+i-1/4},$$

for $1 \leq i \leq \left[\frac{r}{2}\right]$ and $0 < x \leq x_{nn}$.

COROLLARY. The convergence of $Q_n^{(i)}(f;x)$ to $f^{(i)}(x)$ is uniform in every interval $[\varepsilon, A]$, if $\varepsilon, A > 0$.

The detailed proofs will appear in the G. Freud memorial volume of the Journal of Approximation Theory. Some further problems will be treated there about Lagrange interpolation based on the zero and the roots of generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, $\alpha > -1$.

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