КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 154—158

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
SOFIA, 1984, p. 154-158

ON SOME APPROXIMATIONS IN FUNCTION SPACES AND THEIR APPLICATIONS IN NUMERICAL METHODS

Zbigniew Bartoszewski

1. Introduction. In this papers we shall consider certain approximations of a continuous operator mapping the metric space into itself by means of operators transforming finite dimensional spaces of grid functions into themselves as well as approximations of fixed -points of this operator by ε- fixed-points finite dimensional operators.

Let L be a space with the metric d consisting of functions (with real or complex values) defined on the set K compact in some metric space. Let M be some interval $(0,h_0]$, $h_0 > 0$ or a sequence of positive real numbers convergent to zero. Let $\{N_h | h \in M\}$ be a family of grids in K. Let $\{S_h | h \in M\}$ be a family of spaces such that for any $h \in M$ S_h is a space of grid functions (with real or complex values) with the metric d_h. Let $\{r_h | h \in M,\ r_h: L \rightarrow S_h\}$ be a family of restriction operators (however, the restriction of f to N_h need not be equal to $r_h f$). Let $\{p_h | h \in M,\ p_h: S_h \rightarrow L\}$ be the family of extension operators (not necessarily such, that for the grid function y_h the restriction $p_h y_h$ to N_h is equal to y_h). In the sequel, speaking of an arbitrary family of sets U_h or transformations q_h we shall mean the families $\{U_h | h \in M\}$ or $\{q_h | h \in M\}$ respectively. Given the operator $A: L \rightarrow L$. As is well known the finding of solutions of many analitycal problems can be reduced to the finding of fixed-points of the operator A, i.e. such points $x \in L$ that

(1) $$x = Ax \ .$$

However, in many cases we are compelled to confine ourselves to approximate methods of solving equations (1). In these cases instead of one operator A we usually consider a family of operators A_h, $A_h: S_h \to S_h$ and the problem of fixed-points of operators A_h, i.e. the equations

(2) $$x_h = A_h x_h \,.$$

This problem has been studied by many authors, among others by Gorbunov ([2,3]). Gorbunov stipulates (although this stipulation is not mentioned explicitly in [2]) for the operators r_h and p_h to satisfy the condition $r_h p_h x_h = x_h$ for any $x_h \in S_h$. This condition limits greatly the possibilities of choice of operators r_h and p_h. In this paper the stipulation will be replaced by a weaker condition. However, in this case certain changes in some definitions and the introduction of certain others are necessary. Also the main theorem in [2] will have to be changed.

2. Basic definitions. The strong consistency of the family d_h of metrics with the metric d is defined as follows (cf. [2]):

Definition 1. The family of metrics d_h is said to be strongly consistent with the metric d with respect to the family of operators r_h if for any couple of sequences $(x_m)_{m \in N}$, $(y_m)_{m \in N}$ convergent in L

(3) $$d_h(r_h x_m, r_h y_m) \xrightarrow[h \to 0]{} d(x_m, y_m),$$

where the last convergence is uniform with respect to m.

Definition 2. We shall say that the family of operators A_h strongly approximates the operator A with respect to the family of metrics d_h and the operators r_h and p_h if for any sequence (x_h) $(h \in M,\ x_h \in S_h)$ such that the sequence $(p_h x_h)$ is convergent for $h \to 0$ the condition

(4) $$d_h(r_h A p_h x_h,\ A_h x_h) \xrightarrow[h \to 0]{} 0$$

holds.

Definition 3. The operators p_h are said to be approximatively inverse to the operators r_h if for any sequence (x_h) $(h \in M,\ x_h \in S_h)$ such that the sequence $(p_h x_h)$ is convergent for $h \to 0$ the condition

(5) $$d_h(r_h p_h x_h, x_h) \xrightarrow[h \to 0]{} 0,$$

holds.

Proposition 1. If 1° for the family of operators A_h and any sequence (x_h) $(h \in M,\ x_h \in S_h)$ such that the sequence $(p_h x_h)$ converges for $h \to 0$ the condition

(6) $$d_h(A_h r_h p_h x_h, A_h x_h) \xrightarrow[h \to 0]{} 0 \,,$$

2° for any sequence $(x_m)_{m \in N}$ $(x_m \in L)$ convergent in L the condition

(7) $$d_h(r_h A x_m, A_h r_h x_m) \xrightarrow[\substack{h \to 0 \\ m \to \infty}]{} 0$$

holds, then the family of operators A_h strongly approximates the operator A with respect to the family of metrics d_h and the operators r_h and p_h.

To prove the proposition it is sufficient to put $x_m = p_{h_m} x_{h_m}$ and use the triangle inequality.

Remark. If in (7) we put $x_m = x$ for m=1,2,... then we get a condition used by many authors for the definition of the approximation of operator A by the family of operators A_h.

Definition 4. We shall say that the family of spaces S_h and the families of operators r_h and p_h define a convergent approximation of L if for any $x \in L$ the condition

(8) $$d(p_h r_h x, x) \xrightarrow[h \to 0]{} 0$$

holds.

Definition 5. (cf. [2]). We shall say that the family x^h of functions of L is compact in L with $h \to 0$ if for any sequence of functions belonging to this family corresponding to the sequence of values of the parameter h convergent to zero a subsequence convergent in L can be chosen.

3. <u>Main results.</u> We are now able to prove the following

Theorem. If a) Operator A is continuous in L, b) the family of metrics d_h is strongly consistent with the metric d with respect to the family of operators r_h, c) the family of operators A_h strongly approximates A with respect to the family of metrics d_h and the operators r_h and p_h, d) the operators p_h are approximatively inverse with respect to the operators r_h, e) the family of spaces S_h and the families of operators r_h and p_h define a convergent approximation of L, then A possesses at least one fixed-point if and only if there exists a non-negative function $\varepsilon(h)$, $\varepsilon(h) \to 0$ for $h \to 0$ such that the operators A_h possess $\varepsilon(h)$-fixed-points x_h and the family $\{p_h x_h \mid h \in M\}$ is compact with $h \to 0$.

Proof. Necessity. Let x be a fixed-point of A.

Put $\varepsilon(h) = d_h(r_h x, A_h r_h x)$. Then

$$\varepsilon(h) = d_h(r_h Ax, A_h r_h x) \leq d_h(r_h Ax, r_h A p_h r_h x) + d_h(r_h A p_h r_h x, A_h r_h x).$$

It follows from assumptions a), b), e) that the first right-hand component of the last inequality converges to zero with $h \to 0$, whereas assumptions c) and e) imply the convergence to zero with $h \to 0$ of the second right-hand component of the same inequality. Therefore $r_h x$ is a $\varepsilon(h)$-fixed-point of A_h. It follows from assumption e) that the family $p_h r_h x$ is compact with $h \to 0$.

Sufficiency. Assume that for any $h \in M$, x_h is a $\varepsilon(h)$-fixed-point of the operator A_h and the family $p_h x_h$ is compact with $h \to 0$. From the sequence $h \to 0$ we choose a subsequence h_m such that the sequence $(p_{h_m} x_{h_m})_{m \in N}$ is convergent in L. We shall assume in the sequel that $h \to o$ is a sequence of this kind and omit the index m. It is now sufficient to prove, that

$$d(p_h x_h, A p_h x_h) \xrightarrow[h \to 0]{} 0 \, . \tag{9}$$

Put $u_h = r_h p_h$. Then $d(p_h x_h, A p_h x_h) = d_h(u_h x_h, r_h A p_h x_h) +$

$$+ d(p_h x_h, A p_h x_h) - d_h(u_h x_h, r_h A p_h x_h) \, .$$

It follows from assumption b) that

$$d(p_h x_h, A p_h x_h) - d_h(u_h x_h, r_h A p_h x_h) \xrightarrow[h \to 0]{} 0 \, .$$

Next we have the following sequence of inequalities

$$d_h(u_h x_h, r_h A p_h x_h) \leq d_h(u_h x_h, x_h) + d_h(x_h, r_h A p_h x_h) \leq$$

$$\leq d_h(u_h x_h, x_h) + d_h(x_h, A_h x_h) + d_h(A_h x_h, r_h A p_h x_h) \, .$$

It follows from assumption d) that $d_h(u_h x_h, x_h) \xrightarrow[h \to 0]{} 0$.

From assumption the condition

$$d_h(x_h, A_h x_h) \leq \varepsilon(h) \xrightarrow[h \to 0]{} 0$$

holds and from assumption c) it follows that

$$d_h(A_h x_h, r_h A p_h x_h) \xrightarrow[h \to 0]{} 0 \, .$$

Hence condition (9) is satisfied.

It is also easy to prove the following

a) Proposition 2. If a) the formula $d_h(x_h, y_h) = d(p_h x_h, p_h y_h)$ defines a metric in S_h, b) there exists an absolute constant K such that $d(p_h r_h x, p_h r_h y) \leqslant K\, d(x,y)$ and c) the family S_h and the families p_h, r_h define a convergent approximation of the space L, then the family of metrics d_h is strongly consistent with the metric d with respect to the family of operators r_h, the operators p_h are approximatively inverse to the operators r_h and if A is continuous, then also condition (7) holds.

4. Remarks on applications. The Theorem stated here as well as proposition 2 were applied by the authors for the construction of numerical methods of solutions of linear boundary values problems for nonlinear systems of ordinary differential equations belonging to such class of problems which maybe reduced to fixed-point problems of suitably chosen continued operators, defined in the space $L_2^n[a,b]$. The operators r_h were defined by means of Stieklov functions, whereas the operators p_h were defined by means of a functions piecewise constant or a broken lines. The operators A_h was defined by the formula $A_h = \bar{r}_h A p_h$ where $\bar{r}_h : C[a,b] \to S_h$ and $\bar{r}_h x$ is a standart restriction of a continuous function x onto a regular grid N_h on the segment [a,b]. By means of these methods it was possible to solve linear b.v.p. for systems of nonlinear equations unstable with respect to the initial conditions.

References

1. Z.Bartoszewski. Correction and discretisation of general linear b.v.p. for nonlinear normal form systems of ordinary differential equations. (Dissertation, Lomonosow University, Moscow 1982,Russian).
2. A.D.Gorbunov. On some criterion of existence of fixed-points of a continuous operator in functin spaces. Dokl. AN SSSR,227,4,1976, 781-783 (Russian).
3. A.D.Gorbunov. Difference methods for nonlinear problems. In: Voprosy optimizacii vyćislenij - Ob-vo Znanie. Ukr. SSR, Kijew,1977, 3-20 (Russian).

Institute of Mathematics, Gdańsk University
Wita Stwosza 57, 80 952 Gdańsk, Poland