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SOME QUESTIONS ON SPLINE-SOLUTION CONSTRUCTION FOR ORDINARY DIFFERENTIAL EQUATIONS

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Let us consider the Coshy's problem

$$L[y]=f(x,y) \qquad (I)$$

$$y^{(i)}(x_0)=y_0^{(i)} \quad (i=0,1,\dots,n-1), \qquad (2)$$

where L is a linear differential operator with the constant coefficients; $f(x,y)\in KH_y^{(1)}$ in some domain $G\{x_0\le x\le X,\ y_1(x)\le y(x)\le y_2(x)\}$.

The method is the combination of the Voronovskay and the spline-collocation methods. /2/

For every interval $[x_{k-1},x_k]$ $(k=1,2,\dots,m)$ of the arbitrary partition of the integration interval $[x_0,X]$ the initial Chapligin's functions are constructed such that (see, for example, /2/, /5/) $u_{0_k}^{(i)}(x_{k-1})\geqslant v_{0_k}^{(i)}(x_{k-1})=y_0^{(i)}(x_{k-1})$, $u_{0_k}(x)\le y(x)\le v_{0_k}(x)$ and $u_{0_k}(x)\geqslant u_{0_{k-1}}(x)$, $v_{0_k}(x)\le v_{0_{k-1}}(x)$.

In order to construct the approximate spline-solution approximating the solution of (I) - (2), let us replace the function $f(x,y)$ by linear function $z_k=\varphi_k(x)y+\psi_k(x)$ in the domain $G_k\{x_{k-1}\le x\le x_k,\ u_{0_k}(x)\le y(x)\le v_{0_k}(x)\}\subset G$, where $\varphi_k(x)$ and $\psi_k(x)$ are to be defined.

Let on the interval $[x_{k-1},x_k]$

$$\delta_k(x)=\left|f(x,y)-[\varphi_k(x)y+\psi_k(x)]\right|,\quad \rho_k(x)=y(x)-S_k(x). \qquad (3)$$

Then (I) takes the form

$$L[y]=\varphi_k(x)y+\psi_k(x)+\delta_k(x) \qquad (4)$$

and it's solution is $y(x)=S_k(x)+\rho_k(x)$, where $S_k(x)$ is the solution of the linear equation

$$L[S_k]=\varphi_k(x)S_k+\psi_k(x) \qquad (5)$$

consequently $L[y-S_k]=\varphi_k(x)(y-S_k)+\delta_k(x)$.

From here and (3) we receive

$$L[\rho_k]=\varphi_k(x)\rho_k+\delta_k(x). \qquad (6)$$

Let's choose the functions $\varphi_k(x)$ and $\psi_k(x)$ such that

$$\delta_k(\bar{x})=|f(\bar{x},y)-\varphi_k(\bar{x})y-\psi_k(\bar{x})|=\inf_{\varphi,\psi}\max_{u_{0_k}\le y\le v_{0_k}}|f(\bar{x},y)-\varphi(\bar{x})y-\psi(\bar{x})|.$$

Let $A_{1k}(\xi_{1k},\zeta_{1k})$, $A_{2k}(\xi_{2k},\zeta_{2k})\in G_k$ are the strong points of the curve $z=f(\bar{x},y)$, i.e. (see /3/ such that the maximal distance from the strong segment $A_{1k}A_{2k}$ point $R_k(\xi_k,\zeta_k)$ of this curve are between them. Let us draw the segment $B_{1k}B_{2k}$ parallel to $A_{1k}A_{2k}$. Then the surface $z=f(x,y)$ will be placed between the surfaces, formed by the movement of the strong segments.

Let $\varepsilon_k(\bar{x})$ -is the distance between them.

Then $$\varepsilon_k(\bar{x})=\left|\frac{q_k(\bar{x})\cdot\varepsilon_k(\bar{x})-P_k(\bar{x})\cdot\zeta_k(\bar{x})+r_k(\bar{x})}{P_k(\bar{x})}\right|,$$

where $q_k(\bar{x})=\zeta_{2k}(\bar{x})-\zeta_{1k}(\bar{x})$, $P_k(\bar{x})=\xi_{2k}(\bar{x})-\xi_{1k}(\bar{x})$,

$$r_k(\bar{x})=P_k(\bar{x})\cdot\zeta_{1k}(\bar{x})-q_k(\bar{x})\cdot\xi_{1k}(\bar{x}).$$

For all $x\in[x_{k-1},x_k]$ we replace the surface $z=f(\bar{x},y)$ by the surface, formed by the movement of the line

$$z_k=\frac{q_k(\bar{x})}{P_k(\bar{x})}y+\zeta_k(\bar{x})-\frac{q_k(\bar{x})}{P_k(\bar{x})}\xi_k(\bar{x})\pm\frac{1}{2}\varepsilon_k(\bar{x}).$$

This line is the polinom of the best approximation (of y) of order I. Then the error of approximation in G_k will be such that $\delta_k(x)\le\frac{1}{2}\varepsilon_k(x)$.

Thus $\varphi_k(\bar{x})=\dfrac{q_k(x)}{P_k(x)}$, $\psi_k(x)=\zeta_k(x)-\dfrac{q_k(x)}{P_k(x)}\xi_k(x)\pm\dfrac{1}{2}\varepsilon_k(x)$.

Let us replace the surface $z=f(x,y)$ on $[x_{k-1},x_k]$ by the surface formed by the motion of the line

$$z_k=\frac{W_k(\bar{x})}{w_k(\bar{x})}y+\frac{C_k(\bar{x})}{w_k(\bar{x})}\pm\frac{1}{2}|M_k(\bar{x})+m_k(\bar{x})|, \qquad (7)$$

where $w_K(x) = v_{0_K}(x) - u_{0_K}(x)$, $W_K(x) = f(x, v_{0_K}) - f(x, u_{0_K})$,

$$C_K(x) = f(x, u_{0_K}) \cdot v_{0_K} - f(x, v_{0_K}) \cdot u_{0_K}, \quad M_K(x) = \max_{u_{0_K} \le y \le v_{0_K}} \Delta_K(x,y),$$

$$m_K(x) = \min_{u_{0_K} \le y \le v_{0_K}} \Delta_K(x,y), \quad \Delta_K(x,y) = f(x,y) - \left[\frac{W_K(x)}{w_K(x)} y + \frac{C_K(x)}{w_K(x)}\right].$$

Then in the equation (5) $\varphi_K(x) = \dfrac{W_K(x)}{w_K(x)}$, $\psi_K(x) = \dfrac{C_K(x)}{w_K(x)} \pm$

$\pm \frac{1}{2} \left| M_K(x) + m_K(x) \right|$. Moreover

$$\delta_K(x) \le \frac{1}{2} \omega_K(x), \tag{8}$$

where $\omega_K(x) = M_K(x) - m_K(x)$.

Intergrating (5) for example by method from /4/ with initial conditions $S_K^{(i)}(x_{K-1}) = y^{(i)}(x_{K-1})$ $(i = 0, 1, \ldots, n-1,\ K = 1, 2, \ldots, m)$ we find the approximate solution of the problem (I) - (2) in the form of the spline-function $S(x) = S_K(x)$.

Integrating (6) with zero initial conditions $\rho_K^{(i)}(x_{K-1}) = 0$, we obtain the estimation of the solution error on the k-th partition interval. So on the whole interval $[x_0, X]$ the error is $\rho(x) = \max_{1 \le K \le m} \rho_K(x)$.

In particular, for the equation of the first order $y' = f(x,y)$ the linear equation (5) takes the form $S_K'(x) = \varphi_K(x) \cdot S_K + \psi_K(x)$ integrating this equation on $[x_{K-1}, x_K]$ with the initial conditions $S_K^{(i)}(x_{K-1}) = y^{(i)}(x_{K-1})$ $(K = 1, 2, \ldots, m)$ we receive the spline-solution

$$S_K(x) = y_{0_K} \cdot \exp \int_{x_K}^{x} \frac{W_K(t)}{w_K(t)} dt + \int_{x_K}^{x} \left\{ \left[\frac{C_K(t)}{w_K(t)} + \frac{1}{2}\left(M_K(t) + m_K(t)\right) \right] \exp \int_t^{x} \frac{W_K(\tau)}{w_K(\tau)} d\tau \right\} dt$$

Integrating the differential equation $\rho_K'(x) = \varphi_K(x) \rho_K + \delta_K(x)$ with zero initial conditions and taking into consideration (8), we receive

$$\rho_K(f, x) \le \frac{1}{2} \int_{x_K}^{x} \omega_K(t) \cdot \left[\exp \int_t^{x} \frac{W_K(\tau)}{w_K(\tau)} d\tau \right] dt. \tag{9}$$

For the functions of the class $KH^{(1)}$ the following theorem is right.

T h e o r e m. For any function $g(x) \in KH^{(1)}$ on the interval the following inequality is right

$$M - m \le \frac{1}{2KD}\left(K^2 D^2 - d^2\right), \qquad \text{(I0)}$$

where $D = b - a,\ d = g(b) - g(a),\ M = \max\limits_{a \le x \le b} \Delta x,\ m = \min\limits_{a \le x \le b} \Delta x,$

$$\Delta x = g(x) - \ell(g,x),\quad \ell(g,x) = \frac{d}{D}(x-a) + g(a).$$

In the class $KH^{(1)}$ the following function exists

$$g_0(x) = \begin{cases} K(x-a) + g(a), & x \in \left[a, \frac{1}{2}(a+b+dK^{-1})\right), \\ -K(x-b) + g(b), & x \in \left[\frac{1}{2}(a+b+dK^{-1}), b\right], \end{cases}$$

for which the inequality (I0) becomes the equality

$$m_0 = 0,\quad M_0 = \frac{1}{2KD}\left(K^2 D^2 - d^2\right).$$

According this theorem

$$\omega_K(x) \le \frac{1}{2K w_K(x)}\left[K^2 w_K^2(x) - W_K^2(x)\right] = M_{0_K}(x)$$

and consequently

$$\sup_{f \in KH_y^{(1)}} \rho_K(f,x) \le \frac{1}{2}\int_{x_K}^{x} M_{0_K}\left[\exp\int_t^x \frac{W_K(\tau)}{w_K(\tau)}d\tau\right]dt. \qquad \text{(II)}$$

If $z = f(x,y)$ is replaced by the surface formed by the motion of the line

$$z_K = \frac{W_K(\bar{x})}{w_K(\bar{x})}y + \frac{C_K(\bar{x})}{w_K(\bar{x})} + \frac{1}{2}M_{0_K}(\bar{x}) \qquad \text{(I2)}$$

which is the polinom of the best approximation of the function

$$f_{0_K}(\bar{x},y) = \begin{cases} K(y - u_{0_K}) + f(\bar{x}, u_{0_K}), & y \in \left[u_{0_K}, \frac{1}{2}\left(u_{0_K} + v_{0_K} + W_K(x)K^{-1}\right)\right), \\ -K(y - v_{0_K}) + f(\bar{x}, v_{0_K}), & y \in \left[\frac{1}{2}\left(u_{0_K} + v_{0_K} + W_K(x)K^{-1}\right), v_{0_K}\right], \end{cases}$$

which is extrimal on class $KH_y^{(1)}$, then

$$S_{0_K}(x) = y_{0_K}\exp\int_{x_K}^{x}\frac{W_K(t)}{w_K(t)}dt + \int_{x_K}^{x}\left\{\left[\frac{C_K(t)}{w_K(t)} + \frac{1}{2}M_{0_K}(t)\right]\exp\int_t^x \frac{W_K(\tau)}{w_K(\tau)}d\tau\right\}dt$$

and

$$\rho_{0_K}(f,x)=\frac{1}{2}\int_{x_K}^{x} M_{0_K}(t)\cdot\left[\exp\int_{t}^{x}\frac{W_K(\tau)}{w_K(\tau)}d\tau\right]dt,$$

$$\rho(x)=\max_{1\le K\le m}\ \sup_{f\in KH_y''}\ \rho_K(f,x)\le\frac{1}{2}\int_{x_K}^{x} M_{0_K}(t)\left[\exp\int_{t}^{x}\frac{W_K(\tau)}{w_K(\tau)}d\tau\right]dt.$$

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