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SUPERCONVERGENCE IN THE SPLINE-INTERPOLATION

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The splines to be investigated are defined on the real axis R with uniformly spaced nodes at the points $x_j=x_0+jh$ ($j\in Z$), h being a positive number. The space of splines of r-th degree is denoted by S_r . Let F be the class of functions f, which are bounded on every finite interval and there exists a constant n=n(f) such that $f(x)=O(x^n)$ as $x\to\pm\infty$. Some call the functions from F functions of power grouth. We denote by Y the class of sequences $y=(y_j)_{j\in Z}$ for which $y_k=O(k^n)$ as $k\to\pm\infty$ with a constant n=n(y).

Let $\beta\in[0,1)$ be fixed. We define the spline-interpolator $s_r(x)=s_r(f,\beta;x)$ by the following two properties

(1) $\quad s_r(x_j+\beta h)=f(x_j+\beta h)$ for every $j\in Z$,

(2) $\quad s_r\in S_r\cap F$.

This interpolation is wellknown in the case $\beta=(1+(-1)^r)/4$. There are many papers devoted to the existence, uniqueness and estimation of the error in this case (see [1], [2], [3], [4]). The following theorem is a generalization of a theorem of Shoenberg[2].

<u>Theorem 1</u>. Let $\beta\neq(1-(-1)^r)/4$ and $f\in F$. Then $s_r(f,\beta;x)$ exists and is unique.

To estimate the L_p-norm ($\|f\|_p=(\int_{-\infty}^{\infty}|f(x)|^p dx)^{1/p}$ for $1\leq p<\infty$ and $\|f\|_\infty=\text{ess sup}(|f(x)|\,;\,x\in R)$) of the error we need the integral modulus

$$\omega_k(f;h)_p=\sup(\|\Delta_t^k f(.)\|_p\ ;\ 0\leq t\leq h)\ ,$$

where $\Delta_t f(x)=f(x+t)-f(x)$ and $\Delta_t^k f(x)=\Delta_t(\Delta_t^{k-1}f(x))$

and the average modulus

$$\tau_k(f;h)_p=\|\omega_k(f,.;h)\|_p\ ,$$

where $\omega_k(f,x;h)=\sup(|\Delta_t^k f(z)|\ ;\ z,z+kh\in[x-kh/2,x+kh/2]\ )$.

These moduli are investigated in detail in [5]. Here we give some of their properties, from which many consequences of our theorems can be obtained

$$\omega_k(f;h)_p \leqq h\,\omega_{k-1}(f';h)_p \quad ,$$

$$\omega_k(f;h)_p \leqq \tau_k(f;h)_p \quad ,$$

$$\tau_k(f;h)_p \leqq k^2.2^k.4h\,\omega_{k-1}(f';h)_p \quad ; \quad \tau_1(f;h)_p \leqq h\,\|f'\|_p ,$$

$$\tau_k(f;h)_p \leqq kh^k\,\|f^{(k)}\|_p \quad ,$$

$$\tau_1(f;h)_1 \leqq h.Vf \quad , \text{ where } Vf \text{ is the variation of } f .$$

From now on only functions f for which $\tau_k(f;h)_p$ exists, will be considered. We denote the class of these functions by F_k. It is knoun that $f \in F_k$ is equivalent with the existence of two functions $u(x)$ and $v(x)$ such that $u^{(k)}, v^{(k)} \in L_p$ and the inequalities $u(x) \leqq f(x) \leqq v(x)$ hold for all $x \in R$. Obviously $F_k \subset F$.

Remark. It is easy to see that if N is natural number, $h=1/N$ and f is 1-periodic function, then $s_r(f,\beta;x)$ is 1-periodic, too. That is why all our results are in forse for periodic case (if we use the periodic $\tilde{L}_p$-norm).

The estimates of C-norm of the error when r is odd and $\beta=0$ are given by J.H.Ahlberg, E.N.Nilson and J.L.Walsh [6] (see also [3]). They prove that

(3) if $f^{(r)} \in C$, then $\|f^{(k)}-s_r^{(k)}\|_C = o(h^{r-k})$, $k=0,1,\dots,r$;

(4) if $f^{(r)} \in Lip\,\alpha$, then $\|f^{(k)}-s_r^{(k)}\|_C = O(h^{r+\alpha-k})$, $k=0,1,\dots,r$.

S.B.Stečkin and Yu.N.Subbotin [4] proved for arbitrary natural r and $\beta=(1+(-1)^r)/4$ that if $f^{(m)} \in C$, then

(5) $\|f^{(k)}-s_r^{(k)}\|_C \leqq C_1(r,k,m)\,\omega_{r+1-k}(f^{(k)};h)_\infty$

where C_1 is a constant depending only on $r \geqq m \geqq k \geqq 0$.

The estimates of L_p-norm of the error for $r=2$, $\beta=1/2$ and $r=3$, $\beta=0$ is given by A.S.Andreev and V.A.Popov [7]. Let $f \in F_{r+1}$. Then $\|f-s_r\|_p \leqq C_2(r)\,\tau_{r+1}(f;h)_p$.

This estimate is proved for all natural r by author in [8].

Theorem 2. Let $\beta \neq (1-(-1)^r)/4$. Then there exist constants $C_3=C_3(r,\beta)$ and $C_4=C_4(r,\beta)$, depending only on r and β , such that the following estimates hold true

(6) $\|f-s_r\|_p \leqq C_3\,\tau_{r+1}(f;h)_p$ if $f \in F_{r+1}$;

(7) $\|f^{(k)}-s_r^{(k)}\|_p \leqq C_4\,\omega_{r+1-k}(f^{(k)};h)$ if $1 \leqq k \leqq r$ and $f^{(k)} \in F_{r+1-k}$.

All the estimates (3)-(5) follow from this theorem. Here we give two corollaries which one can easy obtain from properties of the average moduli

Corollary 1. Let $0\leq k\leq r$, $k\leq m\leq r+1$ and $f^{(m)}\in L_p$. Then

$$\| f^{(k)}-s_r^{(k)} \|_p=O(h^{m-k}) .$$

Corollary 2. Let $0\leq k\leq m\leq r$ and $f^{(m)}$ is a function of bounded variation. Then $\| f^{(k)}-s_r^{(k)} \|_1=O(h^{m+1-k})$.

The order of h in corollaries cannot be improved, but it is possible in some special points. For example, E.L.Albasiny and W.D.Hoskins[9] proved that if r is odd, ß=0, k is odd and f is a periodic function such that $f^{(r+4)}\in C$, then

$$|f^{(k)}(x_j)-s_r^{(k)}(x_j)| \leq C_5(r,k)h^{r+2-k}(\|f^{(r+2)}\|_C+O(h^2)) \quad , j\in Z.$$

Let $t\in[0,1)$, $0\leq k\leq r$ and $f^{(k)}\in F$. We note

$$y_j=y_j(f,t,k)=f^{(k)}(x_j+th)-s_r^{(k)}(x +th) \quad \text{and} \quad y(f,t,k)=(y_j)_{j\in Z}.$$

As usual $\| y \|_{l_p}=\left\{\sum_{j=-\infty}^{\infty} h.|y_j|^p\right\}^{1/p}$ for $1\leq p<\infty$ and $\| y \|_{l_\infty}=\sup(|y_j| ; j\in Z)$.

The following result is an analog of Theorem 2

Theorem 3. Let $0\leq k\leq r$ and $\beta\neq(1-(-1)^r)/4$. If $f^{(k)}\in F_{r+1-k}$, then $\sup_{t\in[0,1)} \|y(f,t,k)\|_{l_p} \leq C_6 \tau_{r+1-k}(f^{(k)};h)_p$ holds true with a constant C_6 depending only on r and ß .

The order r+1-k of the modulus in the last theorem is an optimal. On the other hand, if we examine the error $y(f,t,k)$ for a specific number $\mu\in[0,1)$, then we can prove a similar estimate with order $r+2-k$ of the modulus. Such an effect is wellknown in the Numerical methods as "superconvergence".

Theorem 4. Let $0\leq k\leq r$, $\beta\neq(1-(-1)^r)/4$ and $f^{(k)}\in F_{r+2-k}$. Then there exists a constant $C_7=C_7(r,\beta)$, depending only on r and ß , such that the inequality

$$\|y(f,\mu,k)\|_{l_p}\leq C_7 \tau_{r+2-k}(f^{(k)};h)_p$$

holds true for $\mu\in[0,1)$ given by $B_{r+1-k}(\mu)=0$ for $k\geq 1$ or $B_{r+1}(\mu)=B_{r+1}(\beta)$ for $k=0$. Moreover, there are no other points $\mu\in[0,1)$ with this property and the order $r+2-k$ cannot be enlarged exept in the trivial case $k=0$, $\mu=\beta$.

Here the Bernoulli polynomials are used and the n-th degree polynomial is denoted by B_n . It follows from the properties of these polynomials that $B_n(t)$ has only two zeros on the interval $[0,1)$ (these are $\mu'=0$ and $\mu''=1/2$ when n is even) and $B_n(t)-B_n(\beta)$ has also two zeros on $[0,1)$ (first of them is $\mu'=\beta$ and, when n is odd, the second is $\mu''=1-\beta$).

Using the properties of the moduli we can obtain many corollaries of Theorem 4 . Here we give two of them

Corollary 3. Let $0 \leqq k \leqq r$ and $f^{(r+2)} \in L_p$. Then

$$\| y(f,\mu,k) \|_{l_p} = O(h^{r+2-k}) \quad .$$

Corollary 4. Let $f^{(r+1)}$ is a function of bounded variation. Then

$$\| y(f,\mu,k) \|_{l_1} = O(h^{r+2-k}) \quad \text{for} \quad k=0,1,\ldots,r \quad .$$

1. Proof of Theorem 1

A linear transformation maps the points $(x_j)_{j \in Z}$ in the integer points Z. Because of that we may assume that the splines from S_r are with nodes at Z and therefore h=1. Let us denote the B-spline of r-th degree with support $(0,r+1)$ by $M_r(x)$, i.e.

$$M_r(x) = \frac{(-1)^{r+1}}{r!} \sum_{j=0}^{r+1} (-1)^j \binom{r+1}{j} (j-x)_+^r \quad ,$$

where $x_+^r = x^r$ for $x \geqq 0$ and $x_+^r = 0$ otherwise. The system

$$\text{(8)} \qquad \sum_{j=0}^{r} M_r(j+1-\beta) y_{k+j} = a_k \qquad (k \in Z)$$

has an important role in our investigation. The polynomial $Q_r(z) = Q_r(\beta;z) = \sum_{j=0}^{r} M_r(j+1-\beta) z^j$ is known as a characteristic polynomial of the sistem (8). It is proved in [4],p.123 that all zeros of $Q_r(z)$ are simple, real and negative. One can futher obtain that $Q_r(-1) \neq 0$ when $\beta \neq (1-(-1)^r)/4$. Let

$$0 < z_1 < \ldots < z_{n-1} < -1 < z_n < \ldots < z_r$$

be the zeros of Q_r and let us denote

$$\text{(9)} \qquad b_k = \sum_{j=1}^{n-1} \frac{z_j^{-k-1}}{Q_r'(z_j)} \quad \text{for} \quad k \leqq -1 \ ; \qquad b_k = \sum_{j=n}^{r} \frac{z_j^{-k-1}}{Q_r'(z_j)} \quad \text{for} \quad k \geqq 0 \ .$$

Obviously there exists a constant $C_8 = C_8(r,\beta)$ for which

$$\text{(10)} \qquad \sum_{k=-\infty}^{\infty} b_k \leqq C_8$$

Using the fact that every nontrivial solution of the homogeneous system $\sum_{j=0}^{r} q_j y_{k+j} = 0$ has an exponential growth when its characteristic polynomial has no zeros of modulus unity, we obtain

Lemma 1. Let $\beta \neq (1-(-1)^r)/4$ and $a=(a_j)_{j \in Z} \in Y$. Then (8) has an unique solution $y=(y_j)_{j \in Z}$ from Y . Moreover $y_k = \sum_{j=-\infty}^{\infty} b_j a_{k-j}$, where b_j are given by (9).

A theorem of Shoenberg (see [1],[2]) gives that every spline from S_r can be represented uniquely as a linear combination of $M_r(j+1-x)$, $j \in Z$. Let $a_k = f(k+\beta)$ and y is the solution from Lemma 1 . Then the spline

$s_r(x)=\sum_{j=-\infty}^{\infty} y_j M_r(j+1-x)$ satisfies (1). Using that M_r has a finite support, we obtain that $y \in Y$ is equivalent with $s_r \in F$. This proves the theorem.

2. Proof of Theorem 3 and Theorem 4

We use the following theorem which is proved by H.Whitney in [10] and [11] (see also [5], p.35).

Theorem A. Let f be a bounded and measurable function in the interval $[a,b]$. Then there exists a polynomial P of degree at most r, for which

$$|f(x)-P(x)| \leqq W(r)\,\omega_{r+1}\left(f,\frac{a+b}{2};\frac{b-a}{r+1}\right) \quad , \quad x \in [a,b]$$

with a constant $W(r)$, depending only on r.

Corollary A. Let A be a linear functional, defined for all functions with bounded k-th derivative on the interval $[a,b]$, and let there exists a constant K such that

$$|A(f)| \leqq K.\sup\left(|f^{(k)}(x)| \; ; \; x \in [a,b] \right) .$$

If $A(P)=0$ for all polynomials P of degree $\leqq r$, then

$$|A(f)| \leqq K.W(r-k)\,\omega_{r+1-k}\left(f^{(k)},\frac{a+b}{2};\frac{b-a}{r+1-k}\right) .$$

We define the linear operator

$$Spl_r(f,\beta;x)=\sum_{j=0}^{r} M_r(j+1-\beta).f(x+jh) - \sum_{j=-\infty}^{\infty} M_r\left(\frac{x_{j+1}-x}{h}\right).f(x_j+\beta h) .$$

It is easy to see that $Spl_r(s,\beta;x)\equiv 0$ for every spline $s \in S_r$. Using (1) we obtain

$$Spl_r(f,\beta;x)=Spl_r(f-s_r,\beta;x)=\sum_{j=0}^{r} M_r(j+1-\beta).(f(x+jh)-s_r(x+jh))$$

and therefore

$$\sum_{j=0}^{r} M_r(j+1-\beta).y_{m+j}(f,t,k)=\frac{d^k}{dx^k} Spl_r(f,\beta;x_m+th) .$$

Applying Lemma 1 for this system, we obtain

$$(11) \qquad y_m(f,t,k)=\sum_{j=-\infty}^{\infty} b_j \frac{d^k}{dx^k} Spl_r(f,\beta;x_{m-j}+th) .$$

On the other hand

$$\frac{d^k}{dx^k} Spl_r(f,\beta;x_m+th) \leqq \sum_{j=0}^{r} M_r(j+1-\beta).\left|f^{(k)}(x_{m+j}+th)\right| + \sum_{j=0}^{r-k} M_{r-k}(j+1-t).\left|\Delta_1^k f(x_{m+j}+\beta h)\right|$$
$$\leqq (2r+2-k)\sup\left(|f^{(k)}(u)| \; ; \; u \in [x_m,x_{m+r+1}]\right) .$$

Using (11), Corollary A, (12), triangular inequality and (10) we receive the following lemma

Lemma 2. If $t \in [0,1)$ is fixed and $\frac{d^k}{dx^k} Spl_r(P,\beta;x_m+th)=0$

for all $m \in Z$ and all polynomials P of degree $\leq n$, then

$$\|y(f,t,k)\|_{L_p} \leq (2r+2-k).W(n+1-k).C_8\, \tau_{n+1-k}(f^{(k)};\frac{n+2}{2(n+1-k)}h)_p .$$

Let us denote $P_n(t)=Spl_r((.)^n,\beta;x_0+th)$ for $t \in [0,1)$. Using that $x^n \in S_r$ for $n \leq r$, we obtain $P_n(t)\equiv 0$ for $n=0,\dots,r$. The proposition of Theorem 3 follows immediately from this and Lemma 2 .

The properties of Bernoulli polynomials give

(12) $\qquad P_{r+1}(t)=B_{r+1}(t)-B_{r+1}(\beta)$ and

(13) $\qquad P_{r+2}(t)=-(r+1)(B_{r+2}(t)-B_{r+2}(\beta))+(r+2)(\frac{r+1}{2}+t+\beta)P_{r+1}(t)$.

It follows from (12) that $P^{(k)}_{r+1}(\mu)=0$ and therefore from Lemma 2 we receive the inequality in Theorem 4 . The fact that there are no other points μ follows from (11) and the fact that $x^{r+1}-s_r((.)^{r+1},\beta;x)$ do not vanish for all x . It is easy to see from (12) and (13) that $P^{(k)}_{r+2}(\mu)=0$ is imposible exept in the case $k=0$, $\mu=\beta$. Hence the order of the modulus cannot be improved. This complete the proof of Theorem 4 .

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