КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 171—172

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 171-172

APPROXIMATION OF BEURLING TEST FUNCTIONS BY ENTIRE FUNCTIONS

Göran Björck

Let ω be a continuous real-valued function on R^n such that

(1) $$0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta) \quad (\forall \xi, \eta \in R^n),$$

(2) $$\int_{R^n} \frac{\omega(\xi)}{1 + |\xi|^{n+1}} d\xi < \infty \text{ , and}$$

(3) $$\log (1 + |\xi|) = O(\omega(\xi)) \quad \text{when } |\xi| \to \infty.$$

For $\phi \in L_1(R^n)$, we consider the Fourier transform $\hat{\phi}(\xi) := \int \exp(-i\langle x,\xi\rangle)\phi(x)dx$. Let $\mathcal{D}_\omega$ be the set of all $\phi \in L_1(R^n)$ such that ϕ has compact support and

$$||\phi||_\lambda := \int |\hat{\phi}(\xi)| \exp(\lambda\omega(\xi)) d\xi < \infty \quad (\forall \lambda > 0).$$

If $\omega(\xi) = \log(1+ |\xi|)$, then $\mathcal{D}_\omega$ will be the Schwartz class $\mathcal{D} = C_o^\infty$, and if $\omega(\xi) = |\xi|^{1/\alpha}$ with $\alpha > 1$, then $\mathcal{D}_\omega$ is a Gevrey class. The classes $\mathcal{D}_\omega$ were created by Beurling [1] and studied by Björck [2]. $\mathcal{D}_\omega$ is an algebra under pointwise multiplication (by Condition (1)), is contained in $\mathcal{D}$ (by(3)), and is non-trivial (by (2)).

The following is an example of approximation-theoretic characterizations of $\mathcal{D}_\omega$ and related spaces, discussed at the conference. The details are published elsewhere (Björck [3]).

For each $c > 0$ and $t > 0$, let $F_\omega(t,c)$ be the set of all entire functions of n complex variables which are the Fourier-Laplace transform of functions of compact support and such that the support of $\hat{f}$ is contained in the set where $\omega \leq t$, and which also satisfy

(4) $$|\hat{f}(\xi)| \leq c\, e^{ct} \quad (\forall \xi \in R^n).$$

Then we have

Theorem. A necessary and sufficient condition for a continuous function u with compact support to be in $\mathcal{D}_\omega$ is that there exists c > 0 such that

$$\inf_{f \in F_\omega(t,c)} \sup_{x \in R^n} |u(x) - f(x)| = O(e^{-\lambda t}) \text{ when } t \to \infty.$$

We finally note that in the special case where $\omega(\xi)$ depends only on $|\xi|$, this theorem deals with approximation by entire functions of exponential type. In fact, let us write $t = \omega_1(s)$, when $\omega(\xi) = \omega_1(|\xi|)$. Then a function is in $\mathcal{D}_\omega$ iff its degree of approximation is $O(\exp(-\omega_1(s))$ when $s \to \infty$, if we approximate by such entire functions of exponential type s which satisfy a condition similar to (4).

References

1. A. BEURLING, Quasi-analyticity and general distributions, Lectures 4 and 5, A.M.S. Summer Institute, Stanford, 1961 (mimeographed).
2. G. BJÖRCK, Linear partial differential operators and generalized distributions, Ark. Mat., 6 (1966), 351-407.
3. ——— Beurling distributions and linear partial differential operators, Symposia Mathematica, Vol. VII (INDAM, Rome 1971), p.367-379, Academic Press, London-New York, 1971.

Department of Mathematics
University of Stockholm
Box 6701 S-113 85 Stockholm Sweden