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# UNIQUENESS THEOREMS FOR GENERALIZED RADON TRANSFORMS

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The theory of the Radon transform [8], [6] tells us how to recover a function defined in the Euclidean plane $E_2$ from its integrals over all straight lines. Here we are interested in the analogous problem where the family of all straight lines is replaced by some fairly general two-parameter family $\Gamma$ of smooth curves in $E_2$. For a series of specific families of curves explicit inversion formulas for the corresponding Radon transform have been given (see e. g. [7], [2], [3]). The far-reaching theories of Helgason [5] and of Gel'fand et al. [4] treat similar problems but are not applicable to general families of curves. For an "arbitrary" family $\Gamma$ of curves it seems to be unknown whether the function $f$ is uniquely determined by the set of data $\{\int_\gamma f\, ds\ ;\ \gamma \in \Gamma\}$, in other words whether

(1) $\quad f \in C_0(E_2)$ and $\int_\gamma f\, ds = 0$ for all $\gamma \in \Gamma$ implies $f = 0$.

Here $C_0(E_2)$ denotes the set of real-valued continuous functions on $E_2$ with compact support, and $ds$ denotes arc length on $\gamma$. Below we will announce a theorem which implies (1) for families $\Gamma$ satisfying certain general conditions.

The set of all oriented straight lines in the plane can be parametrized by the cylinder $M = \mathbb{R}\times S^1$ as follows. For $\omega \in S^1$ let $\omega^\perp$ be the vector obtained by rotating $\omega$ by the angle $\pi/2$ counter-clockwise. Define the mapping $\beta: M\times\mathbb{R} \to E_2$

(2) $\qquad \beta(m,t) = \omega t + p\omega^\perp, \quad t \in \mathbb{R},\ m = (p,\omega) \in M.$

Then the mappings $t \to \beta(m,t)$ describe the set of all (oriented) lines in $E_2$ as $m$ runs through $M$. Since $M\times\mathbb{R}$ is a real-analytic manifold we may speak about real-analytic functions on $M\times\mathbb{R}$. For brevity we shall henceforth say "analytic" instead of "real-analytic". Clearly $\beta$ is analytic. The mapping $\beta$ has the following further properties: (i) $\partial\beta/\partial t \neq 0$ everywhere; (ii) the map $M\times\mathbb{R} \ni (m,t) \to (\beta, \partial\beta/\partial t) \in E_2\times S^1$ is bianalytic; (iii) for any $z \in E_2$ the mapping $\beta^z(\omega,t) = (t + \langle z,\omega\rangle)\omega + \langle z,\omega^\perp\rangle\omega^\perp$ from $S^1\times\mathbb{R}$ onto $E_2$, which describes the subfamily of all lines through $z$, has non-vanishing differential for $t \neq 0$ and all $\omega \in S^1$. These will be the defining properties for the class of mappings $M\times\mathbb{R} \to E_2$ which we will work with.

We shall consider analytic mappings $\alpha: M\times\mathbb{R} \to E_2$. Such a mapping $\alpha$ may be thought of as a family of curves $\alpha(m,\cdot)$, $m \in M$. We shall assume that

(A) for each $m \in M$, $\alpha(m,\cdot)$ is injective and $\partial\alpha/\partial t \neq 0$ for all $t \in \mathbb{R}$.

The associated (unit-)tangent mapping $\tau_\alpha$ from $M\times\mathbb{R}$ into $E_2\times S^1$ will be defined by $\tau_\alpha(m,t) = (\alpha(m,t),(\partial\alpha/\partial t)/|\partial\alpha/\partial t|)$. We shall assume

(B) $\tau_\alpha$ is a bianalytic map from $M\times\mathbb{R}$ onto $E_2\times S^1$.

It follows from (B) that the set of all curves $\alpha(m,\cdot)$, $m \in M$, passing through a given point $z \in E_2$ is a one-parameter family parametrized by the tangent $\omega \in S^1$. Let $\alpha^z(\omega,t)$ be the mapping $S^1\times\mathbb{R} \to E_2$ corresponding to this one-parameter family, each curve being parametrized so that $\alpha^z(\omega,0) = z$ for all $\omega$. In precise terms $\alpha^z(\omega,t) = \alpha(m_\omega,t+t_\omega)$, where $(m_\omega,t_\omega)$ is the unique point such that $\tau_\alpha(m_\omega,t_\omega) = (z,\omega)$. It follows that $\alpha^z$ is an analytic mapping from $S^1\times\mathbb{R}$ into $E_2$. We shall finally assume

(C) for any $z \in E_2$ the map $\alpha^z$ has non-singular differential for $t \neq 0$ and all $\omega$.

This implies that the restriction of $\alpha^z$ to one of the two open sets $S^1 \times \{t \in \mathbb{R};\ t \gtrless 0\}$ is a bianalytic map onto $E_2\setminus\{0\}$. The family $G$ is defined to consist of all analytic mappings $\alpha: M\times\mathbb{R} \to E_2$ satisfying the conditions (A), (B), and (C).

Theorem. Assume $\alpha \in G$, $f \in C_0(E_2)$, and that

$$(3) \qquad \int_{\infty}^{\infty} f(\alpha(m,t))dt = 0 \quad \text{for all } m \in M.$$

Then $f = 0$.

Remark. The assumption that $\alpha \in G$ can be relaxed; it suffices to assume that $\alpha$ satisfies (A), (B), (C) in some neighbourhood of the support of $f$. For instance, in the case of condition (B) this should be interpreted as follows: $\tau_\alpha$ is a bianalytic map from some open subset of $M\times\mathbb{R}$ onto a neighbourhood of supp $f$. (In this case the range of integration in (3) may have to be appropriately restricted.)

As an example we consider the mapping $\alpha(m,t) = \omega v(m,t) + p\,\omega^\perp = \beta(\omega,v(m,t))$, $m = (p,\omega) \in M$, where $v(m,t)$ is an arbitrary analytic function $M\times\mathbb{R} \to \mathbb{R}$ such that $\partial v/\partial t \neq 0$. This means that we have the familiar family of straight lines, but the lines are no longer parametrized by arc length. By appropriate choice of the function $v$ we get the following uniqueness theorem for a Radon transform with "weight functions". Denote the set of all lines in the plane by $L$ and the set of all pairs $(L,x)$ of a line $L$ and a point $x \in L$ by $E(L)$. It is clear that $L$ and $E(L)$ are real-analytic manifolds.

Corollary. Let $\rho(L,x)$ be a positive analytic function on $E(L)$, and assume that $f \in C_0(E_2)$ satisfies

$$\int_L f\, \rho(L,\cdot)ds = 0 \quad \text{for all} \quad L \in \mathcal{L} .$$

Then $f = 0$ .

This result may be of interest in connection with the theory of Emission Computed Tomography; here the emission density plays the role of the function $f$ and the weight function $\rho$ is computed from the (variable) attenuation coefficient [8]

A basic step in the proof of the theorem is the following easy lemma.

Lemma. Assume that $\gamma$ is a proper analytic mapping from $S^1 \times \mathbb{R}$ onto $E_2$ such that

$$\gamma(\omega,t) = t\, b(\omega)\, \omega + O(t^2) \quad , \quad \text{as } t \to 0 ,$$

where $b(\omega) > 0$, and $\gamma$ has non-singular differential outside $t = 0$. Let $d\omega$ denote the arc length measure on $S^1$. Then for $f \in C_0(E_2)$

$$(4) \qquad \int_{S^1} \int_{\mathbb{R}} f(\gamma(\omega,t))dt\, b(\omega)d\omega = \int_{E_2} \left(\frac{2}{|x|} + h(x)\right) f(x)\, dx ,$$

where $h(x)$ is analytic outside the origin and bounded near the origin. More precisely, $h(x)$ can be written $h(x) = H(|x|, x_1/|x|, x_2/|x|)$, where $H$ is analytic.

For an arbitrary fixed $z \in E_2$ we apply the lemma (a trivial modification of it) with $\gamma = \alpha^z$. Since $\alpha^z(\omega,t) = z + t\, b(z,\omega)\omega + O(t^2)$ and $\int f(\alpha^z(\omega,t))dt = 0$ by the assumption, we obtain

$$0 = \int_{S^1} \int_{\mathbb{R}} f(\alpha^z(\omega,t))dt\, b(z,\omega)d\omega =$$

$$= \int_{E_2} \left\{ \frac{2}{|x-z|} + h(z,x-z)\right\} f(x)dx = (Kf)(z)$$

for some function $h(z,y)$ which is analytic for $y \neq 0$; the last equality defines the operator $K$. This operator is an analytic pseudodifferential operator. In fact it belongs to a class of operators studied by Boutet de Monvel and Krée [1]. $K$ is elliptic since its leading part is the convolution with $2/|x|$ (the Fourier transform of this distribution is a constant times $1/|\xi|$ , which does not vanish for any non-zero real $\xi$). For such operators one has a regularity theorem: $f$ is analytic in every open set where $Kf$ is analytic [1, p. 322] . But $Kf = 0$. Hence $f$ is analytic, and since $f$ was assumed to have compact support it must vanish identically.

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