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RATIONAL APPROXIMATION OF ANALYTIC FUNCTIONS

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Let $D \subset \bar{C}$ be an open set, $H(D)$ be the set of the functions holomorphic in D, $E \subset D$ and $K \subset \bar{C} \setminus E$ be compact sets. We denote $F = \bar{C} \setminus D$, and suppose E and F are regular with respect to the Dirichlet problem. If A is a set $\min_A f(z)$ ($\max_A f(z)$) denotes the minimum (the maximum) of the function f on A. Let $g(z,a)$, $a \in E$ be the Green's function of $\bar{C} \setminus E$ with a as a pole. We define a sequence $\{d_n\}$

$$d_n = \sup \min_F \Big(n^{-1} \sum_{k=1}^{n} g(z,z_k) \Big) ,$$

where the sup is taken over all n-tuples $(z_1,z_2,\ldots,z_n)$, $z_k \in K$.

Theorem 1. There exist points $b_{n1},b_{n2},\ldots,b_{nn}$, $b_{nk} \in K$ such that

$$d_n = \min_F \Big(n^{-1} \sum_{k=1}^{n} g(z,b_{nk}) \Big) .$$

Proof. Let $\{z_{1,s}\}$, $\{z_{2,s}\}$,....., $\{z_{n,s}\}$, $s = 1,2\ldots$ $z_{k,s} \in K$, be n sequences such that

$$/1/ \qquad d_n = \lim_{s \to \infty} \min_F \Big(n^{-1} \sum_{k=1}^{n} g(z,z_{k,s}) \Big) .$$

We can assume that $\lim_{s \to \infty} z_{k,s} = b_{nk}$, $b_{nk} \in K$, and hence

$$/2/ \qquad \lim_{s \to \infty} n^{-1} \sum_{k=1}^{n} g(z,z_{k,s}) = n^{-1} \sum_{k=1}^{n} g(z,b_{nk})$$

uniformly on the compact subsets of $(\bar{C} \setminus E) \setminus (\bigcup_{k=1}^{n} \{b_{nk}\})$.

Let us choose the circles $C_1, C_2, \ldots, C_n$ around $b_{n1}, \ldots, b_{nn}$ respectively in such way that

$$/3/ \qquad n^{-1} \sum_{k=1}^{n} g(z, b_{nk}) \geqq N > d_n \,, \quad z \in \bigcup_{k=1}^{n} C_k \,.$$

Then /2/ holds on the boundary of $\bigcup_{k=1}^{n} C_k$, and, because of the superharmonicity, from /3/ follows

$$/4/ \qquad n^{-1} \sum_{k=1}^{n} g(z, z_{k,s}) > d_n \,, \quad s \geqq s_o \,, \quad z \in \bigcup_{k=1}^{n} C_k \,.$$

As for each s

$$\min_F \left(n^{-1} \sum_{k=1}^{n} g(z, z_{k,s}) \right) \leqq d_n \,,$$

we get from /4/ that for $s \geqq s_o$

$$/5/ \qquad \min_F \left(n^{-1} \sum_{k=1}^{n} g(z, z_{k,s}) \right) = \min_{F \setminus (\bigcup_{k=1}^{n} C_k)} \left(n^{-1} \sum_{k=1}^{n} g(z, z_{k,s}) \right)$$

Our statement follows from /1/, /5/ and /2/ .

<u>Theorem 2.</u> The $\lim d_n = d$ exists.

Proof. It is easy to see that d_n is bounded: Let m be the equilibrium distribution of the plain condenser (E,F). Then for the conductor potential

$$u(z) = \int g(z,t)\, dm(t)$$

we have $u(z) \leqq 1/C$, where C is the capacity of (E,F). Let $z_1, z_2, \ldots, z_n$ be points on K. Then

$$\min_F n^{-1} \sum_{k=1}^{n} g(z, z_k) \leqq \int \left(n^{-1} \sum_{k=1}^{n} g(z, z_k) \right) dm(z)$$

$$= n^{-1} \sum_{k=1}^{n} \int g(z, z_k)\, dm(z) = n^{-1} \sum_{k=1}^{n} u(z_k) \leqq 1/C$$

and hence $d_n = 1/C$ for all n.

Let $\underline{d} = \liminf d_n$, $\bar{d} = \limsup d_n$ and let $a > 0$ be an arbitrary number. There exists an integer l such that $\bar{d}-a < d_l$. Let $b_{l1}, b_{l2}, \ldots, b_{ll}$ be such that

$$d_l = \min_F l^{-1} \sum_{k=1}^{l} g(z, b_{lk}) .$$

Let n be an arbitrary integer and let $n = q_n l + r_n$, $0 \leq r_n \leq l-1$. By the definition of d_n we get

$$d_n = \min_F n^{-1} \sum_{k=1}^{n} g(z, b_{lk}) q_n = q_n l d_l n^{-1} \geq (n-r_n) n^{-1} (\bar{d}-a)$$

and hence $\underline{d} = \bar{d}$.

Theorem 3. If $f(z) \in H(D)$, then there exists a sequence $r_n(z)$ of rational functions of degree $\leq n$ and poles on K such that

$$\limsup \| f - r_n \|_E^{1/n} \leq e^{-d} .$$

Proof. We shall follow the Chen's method [1], p. 317. Let

$$V(z_1, z_2, \ldots, z_{n+1}) = \prod_{1 \leq i < j \leq n+1} |z_i - z_j| \Big/ \prod_{i=1}^{n+1} \prod_{j=1}^{n} |z_i - b_{nj}| ,$$

where $z_i \in E$ are arbitrary points, and b_{nj} are the points found in Theorem 1. V is continuous and let $a_{nk} \in E$ be $n+1$ points such that $V(a_{n1}, a_{n2}, \ldots, a_{n,n+1}) = \max V(z_1, z_2, \ldots, z_{n+1})$, $z_i \in E$. Then the function

/6/ $r_{nk}(z) = V(a_{n1}, \ldots, a_{n,k-1}, z, a_{n,k+1}, \ldots, a_{n+1}) / V(a_{n1}, \ldots, a_{n,n+1})$

has the following properties: $|r_{nk}(z)| \leq 1$, $z \in E$, $r_{nk}(a_{nj}) = \delta_{kj}$ where δ_{kj} is the Kronecker symbol.

Let $h(z)$ be the harmonic measure of E with respect to $D \setminus E$, and let us denote by L_s the levelline $L_s = \{ z: h(z) = s \}$, where $s > 0$ is an arbitrarily small, but fixed number. Let $z \in L_{1-s}$ be fixed and consider the function

$$f_n(t) = 1/(z-t) - \sum_{k=1}^{n+1} r_{nk}(t)/(z-a_{nk}) .$$

It has poles only at $t=z$ and $t=b_{nk}$, $k=1,2,\ldots,n$, and zeros at $t = a_{nk}$. If we denote by p_s the distance between E and L_{1-s}, then $|f_n(t)| \leqq (n+2) / p_s$, $t \in L_{1-s}$.

The rational function $f_n(t)(z-t)/w_n(t)$, where

$$w_n(t) = \prod_{k=1}^{n+1} (t-a_{nk}) / \prod_{k=1}^{n} (t-b_{nk})$$

is holomorphic in the hole plain, and hence equal to a constant. Putting $t=z$ we get that this constant is equal to $1/w_n(z)$. Hence $w_n(t)/w_n(z) = f_n(t)(z-t)$. Clearly we can suppose E bounded and the last equality gives

$$/7/ \qquad \max_E |w_n| / |w_n(z)| \leqq (n+2)M_s , \quad z \in L_{1-s} ,$$

where $M_s = (\max |z-t| ,\ z \in L_{1-s},\ t \in E)/p_s$.

The function

$$\sum_{k=1}^{n} g(z,b_{nk}) + \ln(\max_E |w_n|) - \ln |w_n(z)|$$

is harmonic outside L_{1-s}, and from /7/ it follows, for $z \in L_{1-s}$,

$$/8/ \qquad \sum_{k=1}^{n} g(z,b_{nk}) + \ln(\max_E |w_n|) - \ln |w_n(z)| \leqq \ln M_s(n+2) + nN_s ,$$

where $N_s = \{\max g(z,t),\ z \in L_{1-s}\}$. It is clear that $N_s \to 0$ as $s \to 0$. From /8/ we get

$$/9/ \qquad \ln(\max_E |w_n| / \min_{L_s} |w_n|) \leqq \ln M_s(n+2) + nN_s - \min_{L_s} \sum_{k=1}^{n} g(z,b_{nk}).$$

The function $\sum g(z,b_{nk})$ is superharmonic in $D \setminus E$, equal to 0 on E and $\geqq nd_n$ on the boundary of D. Hence, by the two-constant theorem

$$/10/ \qquad \sum_{k=1}^{n} g(z,b_{nk}) \geqq nd_n(1-s) , \quad z \in L_s .$$

From /9/ and /10/ it follows

$$/11/ \qquad \max_E |w_n| / \min_{L_s} |w_n| = (n+2) M_s e^{nN_s - ndn(1-s)} .$$

Let now $f(z) \in H(D)$ and consider the rational function

$$r_n(z) = \sum_{k=1}^{n+1} f(a_{nk}) r_{nk}(z)$$

(cnfer /6/), which is of degree $\leq n$ with poles at $b_{n1}, \dots, b_{nn}$ and $r_n(a_{nk}) = f(a_{nk})$. Then, as $a_{ni} \neq a_{nj}$, $i \neq j$, the function $(f(z)-r_n(z))/w_n(z)$ is holomorphic in D and hence, by the Cauchy formula

$$/12/ \qquad f(z) - r_n(z) = \frac{w_n(z)}{2\pi i} \int_{L_s} \frac{f(t)-r_n(t)}{w_n(t)(t-z)} dt .$$

As the rational function $r_n(t)/w_n(t)$ is holomorphic outside E and has zero at ∞ ,/12/ is equivalent to

$$/13/ \qquad f(z) - r_n(z) = \frac{w_n(z)}{2\pi i} \int_{L_s} \frac{f(t)\, dt}{w_n(t)(t-z)} .$$

Estimating the last itegral in a usual way and using /11/ and the fact $N_s \to 0$ as $s \to 0$ we receive the statement of the theorem.

References

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