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CAUCHY CONVOLUTIONS AND THE MULTIPLIER PROBLEM OF ROOT VECTOR EXPANSIONS

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The present paper deals with some aspects of the multiplier problem of root vector expansion of an operator. Two auxiliary operations (called Cauchy convolutions) for this expansion are systematically used.

Let X be linear topological space (l.t.s.), let $L: X_L \to X$, $X_L \subset X$ be a linear operator in X. By $\varrho(L)$ is denoted the resolvent set of L, by $\sigma_p(L)$ its point spectrum and let $\varrho_0(L) = \{\lambda \in \varrho(L) : (L - \lambda I)(X_L) = X\}$. Let $\Lambda \subset \sigma_p(L)$ be a set of simple eigenvalues (i.e. dim Ker$(L - \lambda I) = 1$ for $\lambda \in \Lambda$) and let the root subspaces $H_\lambda \subset X_L$ be finite dimensional for all $\lambda \in \Lambda$ (i.e. now $H_\lambda = \mathrm{Ker}(L - \lambda I)^{n_\lambda}$ with $n_\lambda = \dim H_\lambda < \infty$). Let also there exists an orthogonal and total system $\{P_\lambda\}_{\lambda \in \Lambda}$ of continuous projections P_λ mapping X onto H_λ and commuting with L (i.e. $P_{\lambda'} P_{\lambda''} = 0$ if $\lambda' \neq \lambda''$ and $P_\lambda f = 0$ for all $\lambda \in \Lambda$ implies $f = 0$). Then for each $f \in X$ one may associate its formal Fourier root vector expansion (f.F.e.) $\mathcal{F} f = \{P_\lambda f\}_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} H_\lambda$. It is clear that the injective mapping $\mathcal{F}: X \longrightarrow \prod_{\lambda \in \Lambda} H_\lambda$ is in general nonsurjective one. We shall denote this correspondence in usual way: $f \sim \sum_{\lambda \in \Lambda} P_\lambda f$. The f.F.e. $\mathcal{F}$ is called simple iff $\dim H_\lambda = 1$ for all $\lambda \in \Lambda$. A root vector system $\{v_0, \ldots, v_{n_\lambda - 1}\}$ in H_λ is called normal iff $Lv_0 = \lambda v_0$ and $Lv_k = \lambda v_k + v_{k-1}$, $1 \le k \le n_\lambda - 1$ if $\dim H_\lambda > 1$. We use the denotation $\mathfrak{X} = \prod_{\lambda \in \Lambda} \mathbb{C}^{n_\lambda}$ and denote its elements by small greek letters: $\alpha = \{\overset{\lambda}{\alpha}_k : 0 \le k \le n_\lambda - 1, \lambda \in \Lambda\}$ and call them "sequences"; $\mathfrak{X}$ is considered with usual Tychonoff topology. By $\mathcal{H}$ is denoted the set of root vector systems in X of the form $\mathcal{U} = \{\overset{\lambda}{u}_k : 0 \le k \le n_\lambda - 1, \lambda \in \Lambda\}$ where $\{\overset{\lambda}{u}_0, \ldots, \overset{\lambda}{u}_{n_\lambda - 1}\}$ is a normal system in H_λ for each $\lambda \in \Lambda$ (we say shortly that $\mathcal{U}$ is normal root vector system in X). If $\{\overset{\lambda}{u}_0, \ldots, \overset{\lambda}{u}_{n_\lambda - 1}\}$ is a normal basis in H_λ for each $\lambda \in \Lambda$ we say shortly that $\mathcal{U}$ is normal basis root vector system in X. It is clear that $\mathcal{H}$ is a linear space with respect to the operations $\mathcal{U} + \mathcal{V} = \{\overset{\lambda}{u}_k + \overset{\lambda}{v}_k\}$, $\alpha \mathcal{U} = \{\alpha \overset{\lambda}{u}_k\}$ for $\alpha \in \mathbb{C}$.

Definition 1. Inner Cauchy convolution yielded by the f.F.e. $\mathcal{F}$ of the operator L is said to be the operation $*_{\mathcal{X}} : \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ defined by the equality $\alpha *_{\mathcal{X}} \beta = \{\sum_{i=0}^{k} \overset{\lambda}{\alpha}_{k-i}\overset{\lambda}{\beta}_i : 0\le k\le n_\lambda-1, \lambda\in\Lambda\}$ for $\alpha = \{\overset{\lambda}{\alpha}_k\}, \beta=\{\overset{\lambda}{\beta}_k\} \in\mathcal{X}$. Outer Cauchy convolution yielded by $\mathcal{F}$ is said to be the operation $*_{\mathcal{H}} : \mathcal{X}\times\mathcal{H}\to\mathcal{X}$ defined by the equality $\alpha *_{\mathcal{H}} \mathcal{U} = \{\sum_{i=1}^{k} \overset{\lambda}{\alpha}_{k-i}\overset{\lambda}{u}_i : 0\le k \le n_\lambda-1, \lambda\in\Lambda\}$ for $\alpha = \{\overset{\lambda}{\alpha}_k\}\in\mathcal{X}$, $\mathcal{U} = \{\overset{\lambda}{u}_k\}\in\mathcal{H}$.

It is clear that for simple f.F.e. the inner Cauchy convolution coincides with the coordinatewise multiplication $\alpha *_{\mathcal{X}} \beta = \{\overset{\lambda}{\alpha}_0\overset{\lambda}{\beta}_0\}_{\lambda\in\Lambda}$ of the"sequences" $\alpha = \{\overset{\lambda}{\alpha}_0\}_{\lambda\in\Lambda}$, $\beta = \{\overset{\lambda}{\beta}_0\}_{\lambda\in\Lambda}$ and the outer Cauchy convolution with the multiplication $\alpha *_{\mathcal{H}}\mathcal{U} = \{\overset{\lambda}{\alpha}_0\overset{\lambda}{u}_0\}_{\lambda\in\Lambda}$ for $\mathcal{U} =\{\overset{\lambda}{u}_0\}_{\lambda\in\Lambda}$.

Lemma 1. a) The inner Cauchy convolution $*_{\mathcal{X}}$ yielded by f.F.e. $\mathcal{F}$ is a continuous,bilinear,commutative and associative operation in $\mathcal{X} = \prod_{\lambda\in\Lambda}\mathbb{C}^{n_\lambda}$ with unit element $\mathcal{E}= \{1,0,\dots,0\}_{\lambda\in\Lambda}$. An element $\alpha\in\mathcal{X}$ is nondivisor of 0 for $*_{\mathcal{X}} \Leftrightarrow \alpha$ is $*_{\mathcal{X}}$-invertible $\Leftrightarrow \overset{\lambda}{\alpha}_0\ne 0$ for $\lambda\in\Lambda$. The inverse $\beta = \{\overset{\lambda}{\beta}_k\} = \alpha^{-1}$ is determined as solution of the set of triangular systems $\overset{\lambda}{\alpha}_0\overset{\lambda}{\beta}_0 = 1$, $\sum_{i=0}^{k}\overset{\lambda}{\alpha}_{k-i}\overset{\lambda}{\beta}_i = 0$, $1\le k\le n_\lambda-1$, $\lambda\in\Lambda$.

b) Outer Cauchy convolution $*_{\mathcal{H}}$ is $\mathcal{X}\times\mathcal{H}\to\mathcal{H}$ bilinear operation and

(1) $$\alpha *_{\mathcal{H}} (\beta *_{\mathcal{H}} \mathcal{U}) = (\alpha *_{\mathcal{X}} \beta) *_{\mathcal{H}} \mathcal{U} \quad \text{for all } \alpha,\beta\in\mathcal{X},\ \mathcal{U}\in\mathcal{H}.$$

c) Let $\mathcal{U}\in\mathcal{H}$ be arbitrary fixed normal basis system in X. Then a system $\mathcal{V}\in\mathcal{H}$ iff there exists $\alpha\in\mathcal{X}$ that $\mathcal{V} = \alpha *_{\mathcal{H}} \mathcal{U}$, i.e. $\mathcal{H} = \mathcal{X} *_{\mathcal{H}} \{\mathcal{U}\}$. A system $\mathcal{V}\in\mathcal{H}$ is normal basis system iff $\mathcal{V} = \alpha *_{\mathcal{H}}\mathcal{U}$ with $*_{\mathcal{X}}$-invertible $\alpha\in\mathcal{X}$; now $\mathcal{U} = \beta *_{\mathcal{H}} \mathcal{V}$ with $\beta\in\mathcal{X}$, $\beta *_{\mathcal{X}}\alpha = \mathcal{E}$.

Further, let $\mathcal{U} = \{\overset{\lambda}{u}_k : 0\le k\le n_\lambda-1, \lambda\in\Lambda\}$ be arbitrary fixed normal basis root vector system, and let $P_\lambda f = \overset{\lambda}{C}_{n_\lambda-1}(f)\overset{\lambda}{u}_0 + \dots + \overset{\lambda}{C}_0(f)\overset{\lambda}{u}_{n_\lambda-1}$ be the representation of the projection P_λ with respect to the normal basis $\{\overset{\lambda}{u}_0,\dots,\overset{\lambda}{u}_{n_\lambda-1}\}$ in H_λ for each $\lambda\in\Lambda$. Let us consider the mapping $\wedge : X\to\mathcal{X}$ defined by the correspondence $f\in X\to\hat{f} = \{\overset{\lambda}{C}_0(f),\dots, \overset{\lambda}{C}_{n_\lambda-1}(f)\}_{\lambda\in\Lambda}\in\mathcal{X}$. It is not difficult to see that $\wedge = \mathcal{J}_{\mathcal{U}}\circ\mathcal{F}$ where $\mathcal{J}_{\mathcal{U}}$ is an algebraical and topological isomorphism between $\prod_{\lambda\in\Lambda} H_\lambda$ and $\mathcal{X} = \prod_{\lambda\in\Lambda}\mathbb{C}^{n_\lambda}$ depending on $\mathcal{U}\in\mathcal{H}$. The mapping $\wedge : X\to\mathcal{X}$ is called Fourier transformation yielded by the operator L with respect to its normal basis root vector system $\mathcal{U}\in\mathcal{H}$; $\hat{f}$ is called Fourier transform of $f\in X$.

Lemma 2. The Fourier transform $\wedge : X\to\mathcal{X}$ of L with respect to a normal basis system $\mathcal{U}\in\mathcal{H}$ is a continuous isomorphical embedding of the l.t.s. X into the space of the "sequences" $\mathcal{X} = \prod_{\lambda\in\Lambda}\mathbb{C}^{n_\lambda}$.

Nevertheless $\wedge$ depends on $\mathcal{U}\in\mathcal{H}$, it is more convenient for use than $\mathcal{F}$ because if $\sim : f\in X\to\tilde{f}\in\mathcal{X}$ is the Fourier transform with respect to a normal basis system $\mathcal{V}\in\mathcal{H}$, then by lemma 1c) one has

(2) $$\tilde{f} = \beta *_{\mathcal{X}} \hat{f}, \quad \hat{f} = \alpha *_{\mathcal{X}} \tilde{f} \quad \text{with } \alpha,\beta\in\mathcal{X},\ \alpha *_{\mathcal{X}}\beta = \mathcal{E}.$$

Definition 2. Let $Y, Z \subset X$ be linear subspaces of X and let $H_\lambda \subset Y$ for each $\lambda \in \Lambda$. An operator $M: Y \to Z$ is said to be (Y,Z)-coefficient multiplier of the f.F.e. $\mathcal{F}: f \sim \sum_{\lambda \in \Lambda} P_\lambda f$ of L iff there exists a normal basis system $\mathcal{U} \in \mathcal{H}$ with corresponding Fourier transformation $\wedge: X \to \mathfrak{X}$ and a "sequence" $\mu \in \mathfrak{X}$ such that

$$(3) \qquad (Mf)^\wedge = \mu *_{\mathfrak{X}} \hat{f} \qquad \text{for each } f \in Y.$$

A "sequence" $\mu \in \mathfrak{X}$ is said to be a (Y,Z)-**multiplier sequence** of $\mathcal{F}$ iff there exists a normal basis system $\mathcal{U} \in \mathcal{H}$ with Fourier transformation $\wedge: X \to \mathfrak{X}$ such that for each $f \in Y$ there is $f^\mu \in Z$ that

$$(3') \qquad (f^\mu)^\wedge = \mu *_{\mathfrak{X}} \hat{f} .$$

From (1),(2) is follows that the"sequence" μ corresponding to M does not depend on the choice of the normal basis system $\mathcal{U} \in \mathcal{H}$ and that μ is uniquelly determined by M. Let $(Y,Z)_{cm}$, $(Y,Z)_{ms}$ denote the linear spaces of all (Y,Z)-coefficient multipliers or multiplier sequences respectively. The easy proved identity $(Y,Z)_{cm} = \{ M \in \mathcal{L}(Y,Z): MP_\lambda = P_\lambda M$ in Y and $ML = LM$ in H_λ for all $\lambda \in \Lambda \}$ gives a characterization of this set independently of the normal basis system. The previous remarks show that for arbitrary normal basis system $\mathcal{U} \in \mathcal{H}$ its Fourier transform $\wedge: X \to \mathfrak{X}$ yields one and the same natural embedding $\vee: \bigcup (Y,Z)_{cm} \to \mathfrak{X}$ (the union is taken on all linear subspaces $Y, Z \subset X$ with $H_\lambda \subset Y$ for all $\lambda \in \Lambda$) such that for each $M \in (Y,Z)_{cm}$ there is unique "sequence" $\mu = \check{M} \in (Y,Z)_{ms}$ that

$$(4) \qquad (Mf)^\wedge = \check{M} *_{\mathfrak{X}} \hat{f} \qquad \text{for each } f \in Y.$$

Also $(Y,Z)^\vee_{cm} = (Y,Z)_{ms}$ and $\vee$ is a linear isomorphism between $(Y,Z)_{cm}$ and $(Y,Z)_{ms}$. We note that these notions are more general than those used in [1] -[5] .

Lemma 3. a) For each linear subspace $Y \subset X$ with $H_\lambda \subset Y$ for $\lambda \in \Lambda$ the space $\hat{Y}$ is without annihilators in $(\mathfrak{X}, *_{\mathfrak{X}})$, i.e. for each $\mu \in \mathfrak{X}$ with $\mu *_{\mathfrak{X}} \hat{f} = 0$ for all $f \in Y$ the equality $\mu = 0$ follows.

b) Let Y,Z,W be linear subspaces of X that $H_\lambda \subset Y, Z$ for each $\lambda \in \Lambda$. If $M \in (Y,Z)_{cm}$, $N \in (Z,W)_{cm}$, then $NM \in (Y,W)_{cm}$ and

$$(5) \qquad (NM)^\vee = \check{N} *_{\mathfrak{X}} \check{M} = \check{M} *_{\mathfrak{X}} \check{N} ;$$

if $\mu \in (Y,Z)_{ms}$, $\nu \in (Z,W)_{ms}$, then $\mu *_{\mathfrak{X}} \nu = \nu *_{\mathfrak{X}} \mu \in (Y,W)_{ms}$.

c) For each linear subspace $Y \subset X$ with $H_\lambda \subset Y$ for all $\lambda \in \Lambda$, the space $(Y,Y)_{ms}$ is commutative subalgebra of $(\mathfrak{X}, *_{\mathfrak{X}})$ and the mapping $\vee$ is isomorphical embedding of the commutative algebra $(Y,Y)_{cm}$ into

$(\mathfrak{X}, *_{\mathfrak{X}})$ such that $(Y,Y)_{cm} = ((Y,Y)_{ms}, *_{\mathfrak{X}})$. If $Y \subset X$ is l.t.s. topologically embedded in X (i.e. the topology τ_Y of Y is weaker or equal to the topology $\tau_X|_Y$ induced by X in Y) and if the closed graph theorem holds in Y, then $\vee$ is a continuous isomorphical embedding of the topological algebra $(Y,Y)_{cm}$ in $(\mathfrak{X}, *_{\mathfrak{X}})$ and $(Y,Y)^{\vee}_{cm} = ((Y,Y)_{ms}, *_{\mathfrak{X}})$.

The previous considerations show that the inner Cauchy convolution $*_{\mathfrak{X}}$ in $\mathfrak{X} = \prod_{\lambda \in \Lambda} \mathbb{C}^{n_\lambda}$ is connected in natural way with f.F.e. $\mathcal{F}$: $f \sim \sum_{\lambda \in \Lambda} P_\lambda f$. The question appears: "Whether there exists an operation $*$ in X that the linear mapping $\wedge : X \longrightarrow \mathfrak{X}$ is a multiplicative one, i.e.

$$(6) \qquad (f * g)^{\wedge} = \hat{f} *_{\mathfrak{X}} \hat{g} \qquad \text{for } f,g \in X."$$

This would mean that $\wedge$ is isomorphical embedding of the algebra $(X, *)$ into the algebra $(\mathfrak{X}, *_{\mathfrak{X}})$. In general this is a complicated problem since the description of X is not a trivial one and it is not clear whether $\hat{X} *_{\mathfrak{X}} \hat{X} \subset \hat{X}$. Theorem 1 below gives an indirect solution of this problem.

Definition 3. A bilinear, commutative and associative operation $*$ in X is said to be a convolution for the operator L in X iff X_L is an ideal of the algebra $(X, *)$ and $L(f * g) = (Lf) * g$ for $f \in X_L$, $g \in X$. Now we say that the algebra $(X, *)$ is yielded by the operator L. Let $Y, Z \subset X$ be linear subspaces of X. An operator $M : Y \longrightarrow Z$ is said to be a (Y,Z)-multiplier of the algebra $(X, *)$ yielded by L iff $Mf * g = f * Mg$ for all $f,g \in Y$. The set of all (Y,Z)-multipliers is denoted by $(Y,Z)_*$.

Theorem 1. Let L have a separately continuous convolution $*$ without annihilators in X and let $P_\lambda \in (X,X)_*$ for all $\lambda \in \Lambda$. Then $\Lambda = \sigma_p(L)$ and there exists a normal basis root vector system $\mathcal{U} = \{\overset{\lambda}{u}_k\}$ that $P_\lambda f = f * \overset{\lambda}{u}_{n_\lambda - 1}$, $f \in X$ and $\overset{\lambda'}{u}_k * \overset{\lambda''}{u}_p = 0$ if $\lambda' \neq \lambda''$; $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = 0$ if $k+p < n_\lambda - 1$, $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = \overset{\lambda}{u}_{k+p-n_\lambda+1}$ if $k+p \geq n_\lambda - 1$ (we call $\mathcal{U}$ "good" normal basis system). If $\wedge$ is its Fourier transformation, then (6) holds. Also $(Y,Z)_{cm} = (Y,Z)_*$ for all linear subspaces $Y, Z \subset X$ with $H_\lambda \subset Y$ for each $\lambda \in \Lambda$.

It is not difficult to prove that for another $\mathcal{V} = \{\overset{\lambda}{v}_k\} \in \mathcal{H}$ with corresponding Fourier transformation $\sim$ there exists $\alpha \in \mathfrak{X}$ such that

$$(7) \qquad (f * g)^{\sim} = \alpha *_{\mathfrak{X}} \tilde{f} *_{\mathfrak{X}} \tilde{g} \qquad \text{for all } f,g \in X.$$

It is suitable to call an operation $*$ in X coefficient convolution of the f.F.e. $\mathcal{F}$ iff $*$ satisfies (7) with some $\alpha \in \mathfrak{X}$ with respect to some $\mathcal{V} \in \mathcal{H}$ and corresponding $\sim : X \longrightarrow \mathfrak{X}$. This is equivalent to the property: " $P_\lambda \in (X,X)_*$ and $*$ is a convolution of L in H_λ for each $\lambda \in \Lambda$."

It is natural to ask what is now the connection between the Fourier transformation $\wedge$ and the Gelfand transformation of the algebra $(X, *)$ yielded by L. If the span of $\bigcup_{\lambda\in\Lambda} H_\lambda$ is dense in X it can be proved that $\overset{\lambda}{C}_0(f)$, $\lambda \in \Lambda$ are all continuous multiplicative linear functionals of $(X, *)$, so now $\wedge$ coincides with Gelfand transformation iff $\dim H_\lambda = 1$ for all $\lambda \in \Lambda$, i.e. iff f.F.e. $\mathcal{F}$ is simple. We note that now $\wedge$ is always injection in contrast of the Gelfand transform when $\dim H_\lambda > 1$ for some $\lambda \in \Lambda$. If this density condition is not true, then the description of the Gelfand transformation is not so evident, but it is also injection if $\dim H_\lambda = 1$ for all $\lambda \in \Lambda$ in contrast of $\wedge$ which is always injection.

The next considerations are proposed to show that the mappings $\wedge$ (with respect to the "good" system), $\vee$ and $*_{\mathfrak{X}}$ naturally appear in the representation of multiplier sequences. In [1] - [3] we proved that by certain assumptions there exists a convolutional isomorphism between a certain space $\mathfrak{M} \subset X$ and $(Y,Z)_* = (Y,Z)_{cm}$ of the form

(8) $J: m \in \mathfrak{M} \longrightarrow J(m) \in (Y,Z)_{cm}$ where $J(m)f = m * f$, $f \in Y$, or

(8') $I_\nu: m \in \mathfrak{M} \longrightarrow I_\nu(m) \in (Y,Z)_{cm}$ where $I_\nu(m) = L_\nu(m * f)$, $f \in Y$

with $L_\nu = L - \nu I$ and fixed $\nu \in \rho_0(L)$. So the next theorem is useful:

<u>Theorem 2.</u> Let $Y, Z \subset X$ be linear subspaces and let $H_\lambda \subset Y$, $\lambda \in \Lambda$. Then:

a) If there exists linear subspace $\mathfrak{M} \subset X$ such that the mapping J is a linear isomorphism between $\mathfrak{M}$ and $(Y,Z)_{cm}$, then $(Y,Z)_{ms} = \widehat{\mathfrak{M}}$.

b) If there exists linear subspace $\mathfrak{M} \subset X$ and $\nu \in \rho_0(L)$ such that the mapping I_ν is a linear isomorphism between $\mathfrak{M}$ and $(Y,Z)_{cm}$, then $(Y,Z)_{ms} = \check{L}_\nu *_{\mathfrak{X}} \widehat{\mathfrak{M}}$, where $\check{L}_\nu = \{\lambda - \nu, 1, 0, \ldots, 0\}_{\lambda\in\Lambda} \in \mathfrak{X}$.

This theorem is useful in a special but important case (see [3]) which is applied in [4], [5]. Following the denotations in [3] let X be l.t.s., let L be a linear operator with <u>convolutional resolvent</u> $R_\lambda f = r(\lambda) * f$, $f \in X$, $\lambda \in \Omega$ ($\Omega \subset \rho_0(L)$ be open) with respect to a separately convolution $*$ in X with $r \in C^\infty(\Omega, X)$. Let Λ be the set of these poles of $r(\lambda)$ which are simple eigenvalues of L (they are always eigenvalues by [3]) and let the corresponding orthogonal system of convolutional projections P_λ, $\lambda \in \Lambda$ mapping X onto H_λ be total in X. In [3] is proved that now $\sigma_p(L) = \Lambda = \{\lambda_k : 1 \le k \le K_L\}$, $K_L \le \infty$ is at most countable set which has no limit points in the open set $\Omega_0 = \Omega \cup \Lambda$ and that $r(\lambda)$ is <u>meromorphic function</u> in Ω_0. Let n_k, $1 \le k \le K_L$ be the multiplicities of the poles λ_k; now $H_{\lambda_k} = \mathrm{Ker}(L - \lambda_k I)^{n_k}$. Then there exists neighbourhoods $W_k \ni \lambda_k$, $W_k \setminus \{\lambda_k\} \subset \Omega$ such that vectorvalued funtions $y_k(\lambda) = -(\lambda-\lambda_k)^{n_k} r(\lambda)$ are extended as $C^{\infty}(W_k, X)$ functions of λ and $\mathcal{U} = \left\{ y_k(\lambda_k), \dots, y_k^{(n_k-1)}(\lambda_k)/(n_k-1)! \right\}_{k=1}^{K_L}$ is a "good" normal basis system in the sense of theorem 1. Let $\wedge$ be its Fourier transformation.

Theorem 3. Let the resolvent R_λ of L be of the form $R_\lambda f = F(\lambda)f + \Psi(\lambda)f.r(\lambda)$, $f \in X, \lambda \in \Omega$ where $F \in C^{\infty}(\Omega_0, \mathcal{L}_0(X,X)), \Psi \in C^{\infty}(\Omega_0, \mathcal{L}_0(X,\mathbb{C}))$ (i.e. $F(\lambda)$ is continuous linear operator for $\lambda \in \Omega_0$ and $F(\lambda)f \in C^{\infty}(\Omega_0, X)$ for each $f \in X$; $\Psi(\lambda)$ is a continuous linear functional in X for $\lambda \in \Omega_0$ and $\Psi(\lambda)f$ is a complexvalued holomorphic function of λ in Ω_0 for each $f \in X$). Let $Y, Z \subset X$ be linear subspaces with $H_{\lambda_k} \subset Y$, $1 \le k \le K_L$. Then:

a) For each $f \in X$ its Fourier transform $\hat{f} \in \mathfrak{X}$ has the representation

$$\hat{f} = \left\{ \frac{1}{s!}\Psi^{(s)}(\lambda_k)f : 0 \le s \le n_k - 1 \right\}_{k=1}^{K_L} = \left\{ \frac{1}{2\pi i}\int_{\Gamma_{\lambda_k}} \frac{\Psi(\lambda)f}{(\lambda-\lambda_k)^{s+1}} d\lambda : 0 \le s \le n_k - 1 \right\}_{k=1}^{K_L} \tag{9}$$

b) If there is $\mathfrak{M} \subset X$ that $(Y,Z)_{ms} = \hat{\mathfrak{M}}$, then $\mu = \{\mu_s^{\lambda_k}\} \in (Y,Z)_{ms}$ iff

$$\mu_s^{\lambda_k} = \frac{1}{s!}\Psi^{(s)}(\lambda_k)m = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_k}} \frac{\Psi(\lambda)m}{(\lambda-\lambda_k)^{s+1}} d\lambda, \ 0 \le s \le n_k - 1, \ 1 \le k \le K_L \text{ with } m \in \mathfrak{M}. \tag{10}$$

c) If there are $\mathfrak{M} \subset X$, $\nu \in \Omega$ that $(Y,Z)_{ms} = \check{L}_\nu *_{\mathfrak{X}} \hat{\mathfrak{M}}$ then $\mu \in (Y,Z)_{ms}$ iff

$$\mu_s^{\lambda_k} = \frac{1}{s!}\frac{\partial^s}{\partial\lambda^s}\left[(\lambda-\nu)\Psi(\lambda)m\right]\Big|_{\lambda=\lambda_k} = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_k}} \frac{(\lambda-\nu)\Psi(\lambda)m}{(\lambda-\lambda_k)^{s+1}} d\lambda, \ 0 \le s \le n_k - 1, \ 1 \le k \le K_L \tag{10'}$$

where $m \in \mathfrak{M}$ and Γ_{λ_k} is a contour enclosing only λ_k of the poles of $r(\lambda)$.

Using these results a complete representation of $(Y,Z)_{cm}$, $(Y,Z)_{ms}$ for Dirichlet and Sturm-Liouville expansions by certain assumptions for Y, Z can be found as it is done in [4], [5] for more special case of scalar coefficient multipliers and scalar multiplier sequences (they are called there coefficient multipliers as well).

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