КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 188—193

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
Sofia, 1984, p. 188-193

ON THE UNIQUENESS OF MONOSPLINES AND PERFECT SPLINES

D. Braess and N. Dyn

1. Introduction. Monosplines and perfect splines of least norm have been investigated in the literature due to their importancce in optimal approximation processes such as numerical quadratures, interpolation schemes and n-widths (see e.g. [14], [3], [12], [11], and references therein).

An algebraic monospline with simple knots is a function of the form

$$M_{\underline{x}}(t) = \int_{x^-}^{x^+} (t-x)_+^k \, h(x)dx - \sum_{i=1}^{n} a_i(t-x_i)_+^k - \sum_{j=0}^{k} c_j t^j \qquad (1.1)$$

with $h(x) \in C[a,b]$ a positive weight function and $\underline{x} \in S^n$, where $S^n := \{\underline{x}=(x^- < x_1 < \ldots < x_n < x^+)\}$. The monospline of least norm is determined by the parameters (a_i, x_i), $i = 1,\ldots,n$, $c_j, j = 0,\ldots,k$, which minimize $\|M_{\underline{x}}(t)\|$.

Similarly a perfect spline with knots $\underline{x} \in S^n$ is defined by

$$P_{\underline{x}}(t) = \int_{x^-}^{x^+} (t-x)_+^k \, h(x) \operatorname{sign} \prod_{i=1}^{n} (x-x_i)\, dx , \qquad (1.2)$$

and that of minimum norm is determined by the knots which minimize $\|P_{\underline{x}}(t)\|$.

During the last decade the problem of the uniqueness of monosplines of least norm and that of the corresponding optimal numerical quadratures has attracted a lot of attention. Uniqueness in case of the sup-norm was shown by Johnson [10], and extended to monosplines with knots of fixed odd multiplicities by Barrar and Loeb [1]. The situation in the L_p-case $1 \leq p < \infty$ is more delicate. It is apparent from counter examples with $h(x) \not\equiv 1$, given by Nürnberger and Braess [13], that for the L_p-norms $1 \leq p < \infty$, additional conditions are necessary to ensure uniqueness.

The dependence of the uniqueness on the form of $h(x)$ became clearer with the result of Barrow, Chui, Smith and Ward [2], who proved uniqueness for the case $p = 2$, $k = 1$ and $h(x)$ log-concave. For $p = 2$, $h(x) \equiv 1$ and general k, uniqueness was proved by Jetter and Lange [9], and then extended by Jetter [8], using a similar method of proof, to the case $p = 1$. For general L_p, $1 < p < \infty$, uniqueness was proved by Bojanov [4].

In a previous paper [5] the authors investigated the uniqueness problem for monosplines and perfect splines defined by extended totally positive (ETP) kernels of the form $K(t,x) = g(t - x)$. In this setting

$$M_{\underline{x}}(t) = \int_{x^-}^{x^+} K(t,x)h(x)dx - \sum_{i=1}^{n} a_i K(t,x_i) , \tag{1.3}$$

and

$$P_{\underline{x}}(t) = \int_{x^-}^{x^+} K(t,x)h(x) \operatorname{sign} \prod_{i=1}^{n} (x - x_i)\, dx . \tag{1.4}$$

These two classes were treated by a unified method, which was applied to the wider class of generalized monosplines. A generalized monospline (GM) with knots $\underline{x}$ of positive multiplicities $\nu_1,\dots,\nu_n$ is defined by

$$\begin{aligned} M_{\underline{x},\underline{\nu}}(t) = {} & \int_{x^-}^{x^+} K(t,x)h(x) \operatorname{sign} \prod_{i=1}^{n} (x - x_i)^{\nu_i} dx \\ & - \sum_{i=1}^{n} \sum_{j=0}^{\nu_i-2} a_{ij} \frac{\partial^j}{\partial x^j} K(t,x_i) , \end{aligned} \tag{1.5}$$

with the convention that $\sum_{j=0}^{-1} \cdot \equiv 0$.

For $\nu_1 = \dots = \nu_n = 1$, the GM in (1.5) is a perfect spline of the form (1.4); for $\nu_1 = \dots = \nu_n = 2$ it is a monospline of the form (1.3). In case of even multiplicities the GM is a multiple-knot monospline.

It was shown in [5] that for a fixed vector of positive multiplicities $\underline{\nu} = (\nu_1,\dots,\nu_n)$, there is a unique set of knots $\underline{x}$ such that the GM (1.5) is of least norm, in case of the L_1 and L_2

norms and $h \equiv 1$. This result was extended to log-concave h in [7].

It is the aim of this paper to present a similar result for L_p-norms $1 < p < \infty$. The method of proof in this case is based on degree theory, as in proof of Bojanov for the algebraic case [4].

2. The Uniqueness Result. In [6], a class of optimal approximation problems were solved via the existence of GM of least monotone norm (norm for which $|f(x)| \leq |g(x)|$ a.e. implies $\|f\| \leq \|g\|$). It was shown in [6] that a GM of least norm is in fact of least norm in a wider class. To state this result, which is central to our analysis, we introduce some notations:

$$K_j(t,x) = \frac{\partial^j}{\partial x^j} K(t,x),$$

$$U_{\underline{x},\underline{\nu}} = \operatorname{span}\{K_j(t,x_i),\ j=0,\dots,\nu_i-1,\ i=1,\dots,n\} , \tag{2.1}$$

$$f_{\underline{x},\underline{\nu}} = \int_{x^-}^{x^+} K(\cdot,x)h(x) \operatorname{sign} \prod_{i=1}^{n} (x - x_i)^{\nu_i} \, dx , \tag{2.2}$$

$$\underline{\nu} - \underline{1} = (\nu_1 - 1,\dots,\nu_n - 1) . \tag{2.3}$$

Theorem 1 [6]: Given positive $h \in C[x^-,x^+]$ and positive multiplicities $\underline{\nu}$,

$$\inf_{\underline{x}\in S^n} \inf_{u\in U_{\underline{x},\underline{\nu}}} \|f_{\underline{x},\underline{\nu}} - u\| = \inf_{\underline{x}\in S^n} \inf_{u\in U_{\underline{x},\underline{\nu}-\underline{1}}} \|f_{\underline{x},\underline{\nu}} - u\| , \tag{2.4}$$

for any monotone norm $\|\cdot\|$.

Moreover the infimum in (2.4) is attained for at least one $\underline{x}^* \in S^n$ and for no boundary point of S^n.

A direct consequence of Theorem 1 is

Corollary 1: For any $\underline{x} \in S^n$, let

$$\hat{M}_{\underline{x},\underline{\nu}} = f_{\underline{x},\underline{\nu}} - u_{\underline{x},\underline{\nu}} = f_{\underline{x},\underline{\nu}} - \sum_{i=1}^{n} \sum_{j=0}^{\nu_i-1} b_{ij}(\underline{x})K_j(\cdot,x_i) , \tag{2.5}$$

with $u_{\underline{x},\underline{\nu}}$ a best approximation to $f_{\underline{x},\underline{\nu}}$ from $U_{\underline{x},\underline{\nu}}$.

Then $\underline{x}^* \in S^n$ is the set of knots of a GM of least norm only if

$$b_{\nu_i-1}(\underline{x}^*) = 0 \quad , \quad i = 1,\dots,n. \tag{2.6}$$

In view of Corollary 1 we define for each $1 \leq p < \infty$ a mapping $T_p : S^n \to R^n$, by

$$(T_p \underline{x})_i = b_{\nu_i - 1}(\underline{x}) \quad , \quad i = 1,\dots,n, \tag{2.7}$$

where the best approximation in (2.5) is taken with respect to the L_p-norm.

Hence, in order to guarantee the uniqueness of the GM of least norm, it is sufficient to analyse the uniqueness of the solution of

$$T_p \underline{x} = \underline{0} \quad , \quad \underline{x} \in S^n. \tag{2.8}$$

This analysis is done by topological degree theory. The application of this theory requires the derivation of several properties of the solution of (2.8). This is done in several lemmas.

Lemma 1 For $p > 1$ and $\underline{x} \in S^n$, let

$$G(x,y) = \int |M_{\underline{x},\underline{\nu}}(t)|^{p-2} K(t,x)K(t,y)\, dt. \tag{2.9}$$

Then the coefficients $b_{ij}(\underline{x})$, $j = 0,\dots,\nu_i - 1$, $i = 1,\dots,n$ in (2.5) are determined implicitly by the conditions

$$F^{(j)}_{\underline{x},\underline{\nu}}(x_i) = 0 \quad , \quad j = 0,\dots,\nu_i - 1 \quad , \quad i = 1,\dots,n, \tag{2.10}$$

where $F_{\underline{x},\underline{\nu}}$ is the GM

$$F_{\underline{x},\underline{\nu}}(x) = \int_{x^-}^{x^+} G(x,y)h(y) \operatorname{sign} \prod_{i=1}^{n} (y-x_i)^{\nu_i}\, dy -$$

$$- \sum_{i=1}^{n} \sum_{j=0}^{\nu_i - 1} b_{ij}(\underline{x}) G_j(x,x_i) \ .$$

Using total positivity arguments, some of which are similar to those in [5], we derived from the characterization in Lemma 1, the important result:

Lemma 2: Let $J_p(\underline{x})$ be the Jacobian of T_p at $\underline{x}$. Then for $1 < p < \infty$ and $\underline{x} \in S^n$ a solution of $T_p \underline{x} = \underline{0}$

$$\operatorname{sign}(J_p(\underline{x}))_{ij} = \varepsilon_i := (-1)^{\sum_{\ell=1}^{i} \nu_\ell}, \ j \neq i, \ i,j=1,\dots,n, \tag{2.11}$$

and

$$\operatorname{sign} \sum_{j=1}^{n} (J_p(\underline{x}))_{ij} = -\varepsilon_i \quad . \tag{2.12}$$

A direct consequence of Lemma 2 is

Lemma 3: Let $T_p \, \underline{x} = \underline{0}$, $1 < p < \infty$. Then

$$(-1)^n \left(\prod_{i=1}^{n} \varepsilon_i \right) \det J_p(\underline{x}) > 0 \tag{2.13}$$

Lemma 4: For a given $q > 1$, there exists $\delta = \delta(q) > 0$, such that any solution of $T_p \, \underline{x} = 0$, $1 \leq p \leq q$, satisfies $\underline{x} \in S^n_\delta$, where

$$S^n_\delta := \{\underline{y} \mid x^- + \delta \leq y_1 < \ldots < y_n \leq x^+ - \delta,$$

$$y_{i+1} - y_i \geq \delta \quad , \quad i = 1, \ldots, n-1\} \quad .$$

The following properties of the solution $T_p \, \underline{x} = \underline{0}$ in case $p = 1$, are derived from the characterization of this solution in [5]:

Lemma 5: There is a unique $\underline{x} \in S^n$ satisfying $T_1 \, \underline{x} = \underline{0}$. For this $\underline{x}$ (2.13) holds with $p = 1$.

The uniqueness of the solution of $T_p \, \underline{x} = \underline{0}$ in S^n for $1 < p < \infty$, is now established by two basic arguments from degree theory [15]:

(i) Let $T_p \in C^1(S^n)$ be a mapping from S^n into R^n. If $T_p \, \underline{x} \neq \underline{0}$ on the boundary of S^n, then the number of solutions of $T_p \, \underline{x} = \underline{0}$ in S^n is finite and

$$\operatorname{degree}\,(T_p, S^n, \underline{0}) := \sum_{T_p \, \underline{x}=0} \operatorname{sign} \det J_p(\underline{x}) \; .$$

(ii) If $T_p \, \underline{x}$ is continuous on $[a,b] \times \overline{S^n}$ and $T_p \, \underline{x} \neq 0$ on the boundary of S^n for all $p \in [a,b]$, then degree $(T_p, S^n, \underline{0})$ is a constant independent of p for $p \in [a,b]$.

These results together with Lemma 3, Lemma 4 and Lemma 5 lead to the conclusion:

Uniqueness Theorem: Let $K(t,x) = g(t - x)$ be ETP and let $h(x) \equiv 1$. Then for $\underline{\nu}$ a vector of positive multiplicities and for p, $1 < p < \infty$, there is a unique GM with knot multiplicity $\underline{\nu}$, which is of least L_p-norm.

The extension of this result to log-concave weight functions h, is still under investigation.

References

1. R.B. Barrar and H.L. Loeb, On monosplines with odd multiplicity of least norm, J. d'Anal. Math. 33, 12-38, 1978.

2. D.L. Barrow, C.K. Chui, P.W.Smith and J.D. Ward, Unicity of best mean approximation by second order splines with variable knots, Math. Comp. 32, 1131-1143, 1978.

3. B.D. Bojanov, Existence of extended monosplines of least deviation, SERDICA Bulg. Math. Pub. 3, 261-272, 1977.

4. B.D. Bojanov, Uniqueness of the monosplines of least deviation, in "Numerische Integration" (G. Hämmerlin, Ed.) ISNM 45, 67-97, Birkhäuser, Basel, 1979.

5. D. Braess and N. Dyn, On the uniqueness of monosplines and perfect splines of least L_1 - and L_2 - norms, J. d'Anal. Math. 41, 217-233, 1982.

6. N. Dyn, Generalized monosplines and optimal approximation, to appear in J. Constructive Approximation.

7. N. Dyn, Uniqueness of least-norm generalized monosplines induced by log-concave weight functions, a preprint.

8. K. Jetter, L_1-Approximation verallgemeinerter konvexer funktionen durch splines mit freien knoten, Math.Z.164,55-66, 1978.

9. K. Jetter and G. Lange, Die eindeutigkeit L_2-optimaler monosplines, Math.Z.158, 23-34, 1978.

10. R.S. Johnson, On monosplines of least deviation, Trans. Amer. Math. Soc. 96, 458-477, 1960.

11. C.A. Micchelli and A. Pinkus, Some problems in the approximation of functions of two variables and n-widths of integral operators, J. Approx. Theory 24, 51-77, 1978.

12. C.A. Micchelli and R.J. Rivlin, A survey of optimal recovery, in "Optimal Estimation in Approximation Theory" (C.A. Micchelli and T.J. Rivlin, Eds.) Plenum Press, 1-54, 1977.

13. G. Nürnberger and D. Braess, Nonuniqueness of best L_p-approximation by spline functions with free knots. Numer. Functional Anal. Optimiz. 4, 199-209, 1981-82.

14. I.J. Schoenberg, Monosplines and quadrature formulae, in "Theory and Applications of Spline Functions" (T.N.E. Greville, Ed.) Academic Press, 157-207, 1969.

15. J.T. Schwartz, Non-linear Functional Analysis, Gordon and Breach, New York-Paris-London, 1969.

Dietrich Braess
Institute fur Mathematik
Ruhr-Universitat
4630 Bochum 1
F.R. Germany

Nira Dyn
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv 69978
Israel