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ASSYMPTOTICAL HOLOMORPHY OF SOME MAPPINGS

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Let $\varphi(x)$ be a twice cotinuously differentiable function, defined in the interval $[0, \infty)$, where $\varphi(x)$ is positive for all $x \in [0, \infty)$. Let us consider the semistrip $S = \{ z = x + iy : x > 0, \ |y| < \varphi(x) \}$ in the z-plane $(z = x + iy)$ Let S_0 belong to the w-plane $(w = u + iv)$ and $S_0 = \{ w = u + iv : u > 0, \ |v| < \frac{\pi}{2} \}$. Let $w = w(z)$ be the conformal (univalent) function which maps the semistrip S onto the cannonical semistrip S_0, so that the triple of points $(0, \pm \varphi(0))$ and $z = +\infty$ are transformed into $(0, \pm \frac{\pi}{2})$ and $w = +\infty$ respectively. The problem that was considered and solved by A.A.Goldberg and T.V.Strocik in their article [1] has the following formulation: Find a mapping $F(z)$: $S \rightarrow S_0$, which should be assymptotically conformal to the mapping $w(z)$, or in other words that $F(z) - w(z)$ uniformly tends to zero when $x \to +\infty$, where some restrictions are imposed on the function $\varphi(x)$. $\varphi(x)$ is said to be the boundary function of S and $w(z)$ is a conformal "strip mapping".

The contribution of the present study is the proof of the assymptotical holomorphy of $F(z)$. The assymptotical holomorphy means that $F(z)$ assymptotically satisfies the Cauchy-Riemann's equations e.g. the expressions:

$$\frac{\partial \operatorname{Re}F(z)}{\partial x} - \frac{\partial \operatorname{Im}F(z)}{\partial y} \quad \text{and} \quad \frac{\partial \operatorname{Re}F(z)}{\partial y} + \frac{\partial \operatorname{Im}F(z)}{\partial x}$$

tend uniformly to zero with respect to y, when $x \rightarrow +\infty$.

It is obvious that an assymptotical conformity does not imply the assymptotical holomorphy (as e.g. there are such functions which tend to zero but their derivatives do not).

Let the boundary function $\varphi(x)$ of S satisfy the following three conditions:

1. The total variation of arc tg$\varphi'(x)$ is bound in the interval $[0, \infty)$;
2. q and x_0 should exist, so that $0 < q < 1$, $x_0 = \text{const} > 0$ and for all $x \geqslant x_0$ the relation holds:

$$\varphi''(x) \geqslant -q \frac{(1 + \varphi'^2(x))(\sqrt{1 + \varphi'^2(x)} + 1)}{\varphi(x)};$$

3. The integral $\displaystyle\int_0^\infty \frac{\varphi \varphi''^2}{1 + |\varphi'|^5} \, dx$ is convergent.

A.A.Goldberg and T.V.Strocik [1] proved the following:

Theorem 1. Let the boundary function $\varphi(x)$ satisfy the conditions 1., 2. and 3. and let $w(z)$ be the conformal strip mapping from S onto S_0. In the case when $x \rightarrow +\infty$ uniformly with respect to y, the following equation holds:

$$w(z) = \lambda + \frac{\pi}{2} \int_{t_0}^{t(x,y)} \frac{\varphi'(s)}{\varphi(s) \operatorname{arctg} \varphi'(s)} \, ds + i\eta(x, y) + o(1)$$

where λ is a suitable real constant, and t and η ($t \geqslant t_0, |\eta| < \frac{\pi}{2}$) represent the unique solution to the system of equations:

$$x = t - \frac{\varphi(t)}{\varphi'(t)} + \frac{\varphi(t)}{\varphi'(t)} \cdot \sqrt{1 + \varphi'^2(t)} \cos\left(\frac{2}{\pi}\eta \operatorname{arctg} \varphi'(t)\right)$$

(1)

$$y = \frac{\varphi(t)}{\varphi'(t)} \cdot \sqrt{1 + \varphi'^2(t)} \sin\left(\frac{2}{\pi}\eta \operatorname{arctg} \varphi'(t)\right)$$

Let us denote the following functions:

(2)
$$F(x,y) = \lambda + \frac{\pi}{2} \cdot \int_{t_0}^{t(x,y)} \frac{\varphi'(s)}{\varphi(s)\,\mathrm{arctg}\,\varphi'(s)}\,ds$$

$g(x,y) = \mathrm{Re}\,F(x,y)$ $\qquad h(x,y) = \mathrm{Im}\,F(x,y)$

$u(x,y) = \mathrm{Re}\,w(z)$ $\qquad v(x,y) = \mathrm{Im}\,w(z)$

$\mathcal{E}^1(x,y) = \mathrm{Re}(w(z) - F(x,y))$ $\qquad \mathcal{E}^2(x,y) = \mathrm{Im}(w(z) - F(x,y))$.

According to Theorem 1, the next equalities hold uniformly with respect to y :

$$\lim_{x\to+\infty} \mathcal{E}^1(x,y) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \mathcal{E}^2(x,y) = 0.$$

The result of the present study is:

Proposition: The mapping $F(x,y)$ is assymptotically holomorphic, i.e.:

$$\lim_{x\to+\infty} \left(\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y}\right) = 0 \qquad \text{and} \qquad \lim_{x\to+\infty} \left(\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right) = 0$$

hold uniformly with respect to y.

Proof: From the definitions of the functions in (2) we obtain:

$$\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} = \mathcal{E}^1_x - \mathcal{E}^2_y \qquad\qquad \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} = -(\mathcal{E}^1_y + \mathcal{E}^2_x).$$

If we derive $g(x,y)$ with respect to x and y we can obtain:

(3)
$$\mathcal{E}^1_x - \mathcal{E}^2_y = \frac{\pi}{2} t_x \frac{\varphi'}{\varphi\,\mathrm{arctg}\,\varphi'} - \eta_y$$
$$\mathcal{E}^1_y + \mathcal{E}^2_x = \frac{\pi}{2} t_y \frac{\varphi'}{\varphi\,\mathrm{arctg}\,\varphi'} + \eta_x \quad .$$

For the partial derivatives of the functions $t(x,y)$ and $\eta(x,y)$ which are implicitly defined by the system (1) the following equalities are valid:

(4) $t_x = \frac{y_\eta}{\Delta}$, $t_y = -\frac{x_\eta}{\Delta}$, $\quad \eta_x = -\frac{y_t}{\Delta}$, $\quad \eta_y = \frac{x_t}{\Delta}$,

where $\Delta = x_t \cdot y_\eta - x_\eta \cdot y_t$

The partial derivatives of the explicit functions $x = x(t,\eta)$ and $y = y(t,\eta)$ which are defined in (1) are:

(5)
$$x_t = \frac{\varphi \cdot \varphi''}{\varphi'^2(1+\varphi'^2)}\left(\sqrt{1+\varphi'^2} - \cos\alpha - \frac{2\eta}{\pi}\sin\alpha\cdot\varphi'\right) + \cos\alpha\sqrt{1+\varphi'^2} \quad ,$$
$$y_t = \sqrt{1+\varphi'^2}\sin\alpha + \frac{\varphi\varphi''}{\varphi'^2\sqrt{1+\varphi'^2}}\cdot\left(\frac{2\eta}{\pi}\varphi'\cos\alpha - \sin\alpha\right) \quad ,$$
$$x_\eta = -\frac{\alpha}{\eta}\sin\alpha\,\frac{\varphi\sqrt{1+\varphi'^2}}{\varphi'} \quad ,$$
$$y_\eta = \frac{\alpha}{\eta}\cos\alpha\,\frac{\varphi\sqrt{1+\varphi'^2}}{\varphi'} \quad ,$$

where $\alpha = \frac{2\eta}{\pi}\,\mathrm{arc\,tg}\,\varphi'$.

Using the condition 2. it can be calculated that:

(6)
$$\Delta \geqslant \frac{|\alpha|}{|\varphi'|}\frac{|\varphi|}{|\eta|}(1+\varphi'^2)(1-q) \quad .$$

Then using (3), (4), (5) and (6) we obtain:

$$(7) \quad |\mathcal{E}^1_x - \mathcal{E}^2_y| \leqslant \frac{\pi}{2}\frac{|\varphi''|}{(1-q)} \frac{\left|\sqrt{1+\varphi'^2} - \cos\alpha - \frac{2\eta}{\pi}\cdot\varphi'\sin\alpha\right|}{\mathrm{arctg}\,\varphi'\,|\varphi'|\,(\sqrt{1+\varphi'^2})^3}$$

$$(8) \quad |\mathcal{E}^1_y + \mathcal{E}^2_x| \leqslant \frac{\pi}{2}\frac{1}{(1-q)} \frac{|\varphi''|}{|\varphi'|} \frac{\left|\frac{2\eta}{\pi}\varphi'\cos\alpha - \sin\alpha\right|}{(\sqrt{1+\varphi'^2})^3}$$

We should point out that from condition 1. follows the existence of the two limits:

(9) $\lim\limits_{x \to +\infty} \varphi'(x) = \gamma$ where γ is a real constant, $0 \leqslant \gamma < \infty$ or $\gamma = +\infty$ and

(10) $\lim\limits_{x \to +\infty} \dfrac{|\varphi''(x)|}{1 + \varphi'^2(x)} = 0.$

Let us estimate the right part of (7) in the following three cases:

1^0 $\delta = 0$. Substituting $\varphi' = \mu$ and applying several times Lopital's rule we get:

$$\lim_{x \to +\infty} \frac{\sqrt{1 + \varphi'^2} - \cos\alpha}{\varphi' \operatorname{arctg} \varphi'} = \lim_{\mu \to 0} \frac{\sqrt{1 + \mu'^2} - \cos \frac{2\eta}{\pi} \operatorname{arctg} \mu}{\mu \operatorname{arctg} \mu} =$$

$$= \frac{\left(\frac{2\eta}{\pi}\right)^2 + 1}{2} = \delta(\eta)$$

where $\frac{1}{2} \leqslant \delta \leqslant 1$.

Consequently,

$$\lim_{x \to +\infty} |\varepsilon^1_x - \varepsilon^2_y| \leqslant \lim_{x \to +\infty} \frac{|\varphi''|}{(1 + \varphi'^2)} \cdot \frac{\delta(\eta)}{\sqrt{1 + \varphi'^2}} = 0$$

2^0 $0 < \gamma < +\infty$. Then obviously

$$\lim_{x \to +\infty} |\varepsilon^1_x - \varepsilon^2_y| = 0$$

3^0 $\gamma = +\infty$. Then:

$$\lim_{x \to +\infty} |\varepsilon^1_x - \varepsilon^2_y| \leqslant \lim_{x \to +\infty} \frac{\pi |\varphi''|}{1 + \varphi'^2} \frac{\left(\sqrt{1 + \frac{1}{\varphi'^2}} - \frac{\cos\alpha}{\varphi'} - \frac{2\eta}{\pi} \sin\alpha\right)}{2|\varphi'|\left(\sqrt{1 + \varphi'^2}\right)^3 \operatorname{arctg} \varphi' (1 - q)} = 0$$

because of (10).

Therefore, we can conclude that:

(11) $\lim\limits_{x \to +\infty} |\varepsilon^1_x - \varepsilon^2_y| = 0$

uniformly with respect to y and independently of the value of γ.

Using (8) and (10) we can see that in each of the cases 1^0, 2^0, 3^0, :

(12) $\lim\limits_{x \to +\infty} |\varepsilon^1_y + \varepsilon^2_x| = 0$

uniformly with respect to y.

From (11) and (12) follows the assymptotical holomorphy of the function F(z) = f(z) + ig(z) which we had to prove.

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References

1. Goldberg, A.A. and T.V.Strocik, Conformal mapping of symmetric half-strips and angular regions, Litovsk, Mat. Sb. 6 (1966), 227-239.