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RECENT PROGRESS IN MULTIVARIATE APPROXIMATION

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This report attempts to survey recent work on the following generic approximation problem:

$$f(s,t) \approx \phi\left(\sum_{i=1}^{n} g_i(s)y_i(t) + \sum_{i=1}^{m} h_i(t)x_i(s) \right)$$

In this equation, f is a bivariate function which we wish to approximate. On the right side, the functions ϕ, g_i, and h_i are prescribed. The functions x_i and y_i are to be chosen to produce a best approximation in some norm. Notice that all the functions on the right side are univariate. Thus we seek to reduce the complexity of f by replacing it with a combination of simpler univariate functions. In its full generality, the problem just posed cannot be solved. Various special cases have been studied with some success. Some of these interesting cases will now be described.

The special case which is easiest to solve is a linear problem in Hilbert space. Let S and T be measure spaces, and suppose that $f \in L_2(S \times T)$. We take the functions $g_1,\dots,g_n$ in $L_2(S)$ and $h_1,\dots,h_m$ in $L_2(T)$. Let ϕ be the identity function: $\phi(r) = r$ for all $r \in \mathbb{R}$. The "coefficient functions" x_i and y_i are sought in $L_2(S)$ and $L_2(T)$ respectively. This problem is solved by an appropriate combination of orthogonal projections. This solution is probably very old, but reference [1] is the earliest I know of. It proceeds as follows. Let P be the orthogonal projection of $L_2(S)$ onto the subspace G spanned by $\{g_1,\dots,g_n\}$. Let Q be the orthogonal projection of $L_2(T)$ onto the subspace H spanned by $\{h_1,\dots,h_m\}$. These two projections are extended to $L_2(S \times T)$ in a standard way. Namely, we define

$$(\bar{P}f)(s,t) = (Pf^t)(s)$$

$$(\bar{Q}f)(s,t) = (Qf_s)(t)$$

where the "sections" f_s and f^t are defined by

$$f_s(t) = f^t(s) = f(s,t)$$

The extended operators are projections of $L_2(S \times T)$ onto the subspaces $G \otimes L_2(T)$ and $L_2(S) \otimes H$, respectively.

Recall that

$$G \otimes L_2(T) = \{ \sum_{i=1}^{n} g_i \cdot y_i \quad : \quad y_i \in L_2(T)\}$$

It is crucial now to verify that $\bar{P}\bar{Q} = \bar{Q}\bar{P}$. Then the general theory of projections [2] shows that $\bar{P}+\bar{Q}-\bar{P}\bar{Q}$ is the orthogonal projection of $L_2(S \times T)$ onto

$$G \otimes L_2(T) + L_2(S) \otimes H$$

All of the preceding construction can be carried out even if G and H are of infinite dimension.

A second special case of our original approximation problem is also rather well understood. Here we assume that S and T are compact Hausdorff spaces, and that $f \in C(S \times T)$. We let G be a subspace of $C(S)$, and we set $H = 0$, ϕ = identity. Thus we are attempting to approximate f by the elements of $G \otimes C(T)$. In the finite-dimensional case, this means to select $y_1, \ldots y_n$ in $C(T)$ to minimize the expression

$$\sup_s \sup_t \left| f(s,t) - \sum_{i=1}^{n} y_i(t) g_i(s) \right|$$

The following theorem is very easy; see [3].

<u>THEOREM</u> <u>If there is a continuous proximity map</u> $A : C(S) \twoheadrightarrow G$ <u>then</u> $G \otimes C(T)$ <u>is proximinal in</u> $C(S \times T)$.

From this theorem it follows that if G is a finite-dimensional Chebyshev subspace in $C(S)$ then $G \otimes C(T)$ is proximinal in $C(S \times T)$. If G is not Chebysheffian, then proximinality can fail. A concrete example has been given in [3].

If we generalize the preceding special case slightly, then we encounter some interesting open problems. First, however, we quote the known positive result from [4].

<u>THEOREM</u> <u>Let</u> G <u>and</u> H <u>be finite-dimensional subspaces in</u> $C(S)$ <u>and</u> $C(T)$ <u>respectively. If</u> G <u>has a continuous proximity map and</u> H <u>has a Lipschitzian proximity map, then</u> $G \otimes C(T) + C(S) \otimes H$ <u>is proximinal in</u> $C(S \times T)$.

This theorem does not encompass the case $G = \Pi_n$ and $H = \Pi_m$ if $\min\{n,m\} \geq 1$. The simplest open question here is whether $\Pi_1 \otimes C(T) + C(S) \otimes \Pi_1$ is proximinal, taking $S = T = [0,1]$. Thus we do not know whether the approximation problem

$$f(s,t) \approx y_1(t) + s y_2(t) + x_1(s) + t x_2(s)$$

has a minimal solution with continuous functions x_i, y_i. See [5] and [6] for further results on these problems.

The preceding theorem does not possess an algorithmic proof except in the case when G and H are the subspaces of constant functions in C(S) and C(T). In this case, the algorithm of Diliberto and Straus provides a construction of a best approximation. Both proximity maps are nonexpansive.

Substantial progress has recently been made by von Golitschek in the area of algorithmic proofs of proximinality. Here is one of his results [7].

THEOREM Assume that $f \in C(S \times T)$, $g \in C(S)$, $g > 0$, $h \in C(T)$, $h > 0$, $\phi \in C(\mathbb{R})$, and $\phi^{-1} \in C(\mathbb{R})$. Then f has a best approximation of the form

$$f(s,t) \simeq \phi(x(s)h(t) + y(t)g(s))$$

with $x \in C(S)$ and $y \in C(T)$.

The von Golitschek algorithm requires these two functions:

$$k(s,t) = \phi^{-1}(f(s,t) - \alpha) \;/\; g(s)h(t)$$

$$K(s,t) = \phi^{-1}(f(s,t) + \alpha) \;/\; g(s)h(t)$$

Here α is a parameter which, ideally, would be set equal to the minimal deviation ρ between f and $\phi\,(xh + yg)$. If ρ is not known, then α is any estimate of ρ. The algorithm will then disclose whether $\alpha < \rho$, $\alpha = \rho$, or $\alpha > \rho$. The calculations to be performed are these, for $n = 0,1,2,\ldots$

$$x_o(s) = 0 \qquad\qquad y_o(t) = \inf_s K(s,t)$$

$$x_n(s) = x_{n-1}(s) \vee \sup_t \; [k(s,t) - y_{n-1}(t)]$$

$$y_n(t) = y_{n-1}(t) \wedge \inf_s \; [K(s,t) - x_n(s)]$$

The symbols $\vee$ and $\wedge$ denote the pointwise maximum and minimum, respectively. Concerning this algorithm, von Golitschek has proved the following results.

THEOREM If $\alpha > \rho$ then there is an index n for which $y_n = y_{n-1}$. When this occurs, $\|f - \phi \circ (x_n gh + y_n gh)\| \leq \alpha$. If $\alpha = \rho$ then the sequences $\{x_n\}$ and $\{y_n\}$ converge uniformly and monotonically to functions x and y satisfying $\|f - \phi \circ (xgh + ygh)\| = \rho$. The inequality $\alpha < \rho$ holds if and only if $\inf_t y_n(t) < -2\|K\| - \|k\|$ for some n.

The algorithm can also be applied to $f \in C(D)$ for some domains $D \subset S \times T$. Certain technical conditions must be placed on D, however. The reader is referred to [7]. The fact that C(D) and $C(S \times T)$ are quite different in such approximations was noted already in the work of Ofman [8].

Another of the very interesting Banach spaces for approximation theory is $L_1(S)$, the space of integrable functions on a measure space S. If S and T are measure spaces, we may wish to approximate an $f \in L_1(S \times T)$ by combinations of univariate L_1-functions. Here is an existence theorem.

THEOREM Let S and T be finite measure spaces. Let $f \in L_1(S \times T)$ and let H be a reflexive subspace of $L_1(T)$. Then f has a best approximation in $L_1(S) \otimes_\gamma H$.

In this theorem, γ is the tensor norm which makes $L_1(S \times T)$ equal to $L_1(S) \otimes_\gamma L_1(T)$. In the finite-dimensional case, the theorem asserts the existence of $x_i \in L_1(S)$ to minimize

$$\int\int \left| f(s,t) - \sum_{i=1}^{n} h_i(t)x_i(s) \right| ds\, dt$$

The above theorem is a special case of the following recent result of Khalil.

THEOREM If H is a reflexive subspace of a Banach space Y, and if S is a finite measure space, then $L_1(S,H)$ is proximinal in $L_1(S,Y)$.

The space $L_1(S,Y)$ consists of all Bochner integrable functions on the measure space S with values in the Banach space Y. For a systematic exposition of such spaces, see [10].

For function spaces which are conjugate spaces, the weak* topology is available to assist in proving results about proximinality. Here is a typical theorem that is proved using weak* compactness. See [4].

THEOREM Let G and H be finite-dimensional subspaces in conjugate Banach spaces X^* and Y^*. Then

$$X^* \otimes H + G \otimes Y^*$$

is complemented and proximinal in $X^* \otimes_\lambda Y^*$.

Here λ is the injective cross norm; i.e., the norm which elements of the tensor product receive when interpreted as linear operators. It is also the norm which makes the equation $C(S \times T) = C(S) \otimes_\lambda C(T)$ true. A corollary of the preceding theorem is this:

THEOREM Let S and T be σ-finite measure spaces. Let G and H be finite-dimensional subspaces in $L_\infty(S)$ and $L_\infty(T)$ respectively. Then

$$L_\infty(S) \otimes H + G \otimes L_\infty(T)$$

is proximinal in $L_\infty(S \times T)$.

A similar theorem is of course true for $\ell_\infty(S \times T)$.

The first theorem cited in this report, concerning $L_2(S \times T)$, has the following generalization. See [4].

THEOREM Let G and H be subspaces having linear proximity maps in Banach spaces X and Y respectively. Then for any uniform cross norm α on $X \otimes Y$, the subspace

$$X \otimes_\alpha H + G \otimes_\alpha Y$$

has a linear proximity map, and so is proximinal in $X \otimes_\alpha Y$.

We conclude this brief survey with an elegant result for the space $L_1(S \times T)$. See [11].

THEOREM Let S and T be finite measure spaces. Let G and H be finite-dimensional subspaces in $L_1(S)$ and $L_1(T)$ respectively. Then the subspace

$$L_1(S) \otimes H + G \otimes L_1(T)$$

is proximinal in $L_1(S \times T)$.

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