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SPLINE BASES IN CLASSICAL FUNCTION SPACES ON COMPACT C^∞ MANIFOLDS. PART III

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Introduction to Part III. The frame work is the same as in Parts I and II of the joint work with T.Figiel [3]. On a d-dimensional compact C^∞ manifold M we are given measure μ, finite, smooth and locally equivalent to Lebesgue measure. Our purpose is to prove the stated below Theorem C which is an essential improvement of both Theorems A and B [3]. More exactly, since the basis in Theorem C is the one constructed in the proof of Theorem B, Part I, [3], we need to prove only (C.1),(C.2),(C.4) and the corresponding statements (C.1),(C.2), (C.4). For the discussion of the dualities in the Sobolev and Besov spaces we refer to [4].

Given a numerical sequence $a=(a_n)_{n=1}^\infty$ and parameters $-\infty<\rho<\infty$, $1\le p,q\le\infty$ we let

$$\|a\|_{b^\rho_{p,q}}=\Big\{\sum_{\mu=0}^{\infty}\Big[2^{\mu\rho}\Big(\sum_{n=2^\mu}^{2^{\mu+1}-1}|a_n|^p\Big)^{1/p}\Big]^q\Big\}^{1/q}.$$

Theorem C. Let M,μ and the integer $m\ge 0$ be given. Then, there are functions $\varphi_n\in C^m(M)$, $\mathring{\varphi}_n\in C^m(M)$, $n=1,2,\ldots,$ such that

(C.0) $$(\varphi_i,\mathring{\varphi}_j)_{L_2(\mu)}=\delta_{i,j},\quad i,j=1,2,\ldots$$

(C.1) (φ_n) is a Schauder basis in $C^k(M)$, $W^k_p(M)$, $1\le p<\infty$, for each integer k: $|k|\le m$.

(C.2) (φ_n) is an unconditional Schauder basis in $W^k_p(M)$, $1<p<\infty$, $|k|\le m$.

(C.3) (φ_n) is an unconditional Schauder basis in each separable Besov space $B^s_{p,q}(M)$ with $|s|<m$, $1\le p,q\le\infty$, and

$$\|a\|_{b^\rho_{p,q}}C^{-1}\le\Big\|\sum_{n=1}^{\infty}a_n\varphi_n\Big\|_{B^s_{p,q}(M)}\le C\,\|a\|_{b^\rho_{p,q}},$$

where $\rho=\frac12-1/p+s/d$ and $C<\infty$ depends on m,s and the manifold.

(C.4) Let for given integers $k,l:-m\leq l<k\leq m$

$$R_{k,l}f = \sum_{n=1}^{\infty} n^{\frac{k-l}{d}} a_n \varphi_n , \quad a_n=(f,\varphi_n)_{L_2(\mu)}.$$

Then, for $1<p<\infty$, $R_{k,l}:W^k_p(M) \longrightarrow W^l_p(M)$ is a linear isomorphism.

Remark. Theorem C has its dual version Theorem $\mathring{C}$ wchich can be obtained formally by replacing $(C,I),W^k_p,B^s_{p,q},C^k,R_{k,l},\varphi_n,\mathring{\varphi}_n$ by $(\mathring{C},I)$, $\mathring{W}^k_p,\mathring{B}^s_{p,q},\mathring{C}^k,\mathring{R}_{k,l},\mathring{\varphi}_n,\varphi_n$, respectively.

12. Regular ordering and bases in the standard Sobolev spaces.

In dimension d for the sets $Z_1,\ldots,Z_d$, $Z_j\subset\partial I$, $I=\langle 0,1\rangle$, the complementary sets $Z,Z'\subset\partial I^d$ are defined as in (2.37),I and (2.47),I *), respectively. For the fixed order of smootheness m define $n(Z_j)$ like in Definition 9.1,II. We are interested in introducing suitable orderings in the index sets $N^d(Z)=N(Z_1)\times\ldots\times N(Z_d)$ where $N(Z_j)=N_+ + n(Z_j)$, $N_+=(1,2,\ldots)$, and in $N^d_n(Z)=N_n(Z_1)\times\ldots\times N_n(Z_d)$ where $N_n(Z_j)=\{i\in N(Z_j):i\leq n\}$. This notation is related to that one introduced in Section 10,II, as follows:

$N^d(Z)=\{\underline{n} \geq \underline{n}(Z)\}$, $N^d_1(Z)=N_o$ and $N^d_{2^\mu}(Z)\setminus N^d_{2^{\mu-1}}(Z)=N_\mu$ for $=1,2,\ldots$

Both these notations will be used later on. For the ordering we are going to define by induction in d a family $\{\varphi(\cdot;Z,d),Z\subset\partial I^d\}$ of 1-1 and onto mappings $\varphi(\cdot;Z,d):N^d(Z) \longrightarrow N_+$ such that

(12.1) $\varphi(\cdot;Z,d)=\varphi(\cdot;Z',d)$,

(12.2) $\varphi(N^d_n(Z);Z,d)=N_+\cap(1,\ldots,\nu_n)$, $\nu_n=\#N^d_n(Z)$ for $n\geq 2$,

(12.3) $\#\varphi(N^d_n(Z);Z,d)=\varphi(\underline{n};Z,d)$ for $\underline{n}=(n,\ldots,n)$ and $n\geq 2$.

The construction is being carried out in three steps.

First step. For d=1 we have $Z=Z_1$ and define $\varphi(n;Z,d)=n-n(Z_1)$ for $n\in N(Z_1)$. It then follows that (12.1)-(12.3) are satisfied.

Second step. Assume that for $d\leq k$, with fixed k, the families $\{\varphi(\cdot;Z,d),Z\subset\partial I^d\}$ satisfying (12.1)-(12.3) are given.

Third step. The construction of $\varphi(\cdot;Z,k+1)$ for given $Z\subset\partial I^{k+1}$ we start by choosing an arbitrary independent of Z order in $\{e\subset\tilde{D}:e\neq\emptyset\}$, $\tilde{D}=\{1,\ldots,k+1\}$, such that $\{e\subset\tilde{D}:e\neq\emptyset\}=\{e_1,\ldots,e_K\}$, $e_K = D$, $K=2^{k+1}-1$.

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*) Here and later on the Roman I,II refer to the corresponding Part of [3].

Then, by induction in $n \geq 2$, we will define 1-1 and onto mappings $\varphi_n(\cdot;Z,k+1):N_n^{k+1}(Z) \longrightarrow N_{\nu_n}$ where $\nu_n = \# N_n^{k+1}$.

The mapping $\varphi_2(\cdot;Z,k+1)$ can be taken arbitrary modulo the conditions: $\varphi_2(\cdot;Z,k+1)=\varphi_2(\cdot;Z',k+1)$ and $\varphi_2(N_2^{k+1}(Z);Z,k+1)=\varphi_2((2,\dots,2);Z,k+1)$.

The definition of $\varphi_{n+1}(\cdot;Z,k+1)$ will be compatible with the following decomposition

$$N_{n+1}^{k+1}(Z) = N_n^{k+1}(Z) \cup \bigcup_{i=1}^{K} N_{n+1,e_i}^{k+1}(Z),$$

where

$$N_{n+1,e}^{k+1}(Z) = \left\{\underline{n} \in N_{n+1}^{k+1}(Z) : n_j \leq n \text{ for } j \in \tilde{D} \setminus e,\ n_j = n+1 \text{ for } j \in e\right\},$$

and $\tilde{D} = \{1,\dots,k+1\}$. Now,

$$\varphi_{n+1}(\underline{n};Z,k+1) = \begin{cases} \varphi_n(\underline{n};Z,k+1) & \text{for } \underline{n} \in N_n^{k+1}(Z), \\ \psi_i(\underline{n}) & \text{for } \underline{n} \in N_{n+1,e_i}^{k+1}(Z),\ i=1,\dots,K; \end{cases}$$

where with $d_j = \#(\tilde{D} \setminus e_j)$ and $Z^{(j)}$ being the boundary set in I^{d_j} corresponding to $(Z_\nu, \nu \in e_j)$ we have

$$\psi_i(\underline{n}) = \varphi_n((\underbrace{n,\dots,n}_{k+1});Z,k+1) + \sum_{j=1}^{i-1} \varphi((\underbrace{n,\dots,n}_{d_j});Z,k+1) + \varphi(\underline{n};Z^{(i)},d_i).$$

We now define $\varphi(\cdot;Z,k+1)$ by the formula

$$\varphi(\underline{n};Z,k+1) = \begin{cases} \varphi_2(\underline{n};Z,k+1) & \text{for } \underline{n} \in N_2^{k+1}(Z) \\ \varphi_n(\underline{n};Z,k+1) & \text{for } \underline{n} \in N_n^{k+1} \setminus N_{n-1}^{k+1}(Z),\ n \geq 3. \end{cases}$$

This completes the construction of the desired family $\varphi(\cdot;Z,d), Z \subset \partial I^d$.

Definition 12.4. Let $d \geq 1$ and let the boundary set $Z \subset \partial I^d$ be given as in (2.37), I. Then the sequence $(a_n)_1^\infty$ is said to be a regular ordering of $(a_{\underline{n}}, \underline{n} \in N^d(Z))$ if there is $\varphi(\cdot;Z,d)$ as constructed above such that $a_n = a_{\underline{n}}$ for $n = \varphi(\underline{n};Z,d)$.

We refer for the definition of $(F_{\underline{n}}^{(m)}(\cdot;Z), \underline{n} \geq \underline{n}(Z))$ and $(G_{\underline{n}}^{(m)}(\cdot;Z), \underline{n} \geq \underline{n}(Z))$ to Sections 9, II and 10, II, respectively. The corresponding regular orderings are denoted below by $(F_n^{(m)}(\cdot;Z))_1^\infty$, $(G_n^{(m)}(\cdot;Z))_1^\infty$. The properties (9.10), II and (10.20), II imply

$$(F_n^{(m)}(\cdot;Z'),F_{n'}^{(m)}(\cdot;Z))_{L_2(Q)} = \delta_{n,n'}$$

(12.5)

$$(G_n^{(m)}(\cdot;Z'),G_{n'}^{(m)}(\cdot;Z))_{L_2(Q)} = \delta_{n,n'}$$

where $Q=I^d$. It also follows that for $\mu=0,1,\dots$

(12.6) $\operatorname{span}[\, F_{\underline{n}}^{(m)}(\cdot;Z),\ \underline{n}\in N^d_{2^\mu}(Z)\,] = \operatorname{span}[\, G_{\underline{n}}^{(m)}(\cdot;Z),\ \underline{n}\in N^d_{2^\mu}(Z)\,]$,

or else by (12.3) with $n=2^\mu$ we get

(12.7) $\operatorname{span}[\, F_j^{(m)}(\cdot;Z),\ j=1,\dots,\nu_n] = \operatorname{span}[\, G_j^{(m)}(\cdot;Z), j=1,\dots,\nu_n]$,

where $\nu_n = \# N_n^d(Z)$.

Theorem 12.8. Let $d\in N_+$, $m\in N$, $Q=I^d$, and let Z be a boundary set given as in (2.37),I. Then both $(F_n^{(m)}(\cdot;Z))_1^\infty$ and $(G_n^{(m)}(\cdot;Z))_1^\infty$, are Schauder bases in all the following spaces: $C^k(Q)_Z$, $W_p^k(Q)_Z$, $1\le p<\infty$, $k=0,\dots,m$; $B^s_{p,q}(Q)_Z$, $0<s<m$, $1\le p,q\le\infty$. Moreover, the norms of the partial sums are bounded uniformly in p and q.

Proof. We may restrict the proof to the Sobolev case only. For each of the the two systems the proof is different.

Define

$$S_\nu^{(m)}(f;Z) = \sum_{j=1}^{\nu} (f,F_n^{(m)}(\cdot;Z'))F_n^{(m)}(\cdot;Z),$$

then for $\nu_n = \# N_n^d(Z)$, $n\ge 2$

$$S_{\nu_n}^{(m)}(f;Z) = \sum_{\underline{n}\in N_n^d(Z)} (f,F_{\underline{n}}^{(m)}(\cdot;Z'))F_{\underline{n}}^{(m)}(\cdot;Z).$$

Since $N_n^d(Z)$ is a product set it follows by the results of Section 9,II, that for some $C=C(m,d)<\infty$ the following inequality

(12.9) $\|S_\nu^{(m)}(f;Z)\|_p^{(k)}(Q) \le C\ \|f\|_p^{(k)}(Q)$, $f\in W_p^{(k)}(Q)_Z$,

holds for $\nu=\nu_n$, $n\ge 2$ and $k=0,\dots,m$. This and induction in d parallel to the one used in the definition of $\varphi(\cdot;Z,d)$ imply that (12.9) holds for every $\nu\in N_+$.

In the second case define

$$T_\nu(f;Z) = \sum_{j=1}^{\nu}(f,G_j^{(m)}(\cdot;Z'))G_j^{(m)}(\cdot;Z).$$

The equality (12.7) implies for some $C=C(m,d)<\infty$ and for $n=2^\mu$, $\mu=1,2,\dots$; $k=0,\dots,m$ the following inequality

(12.10) $\| T^{(m)}_{\nu_{2n}}(f;Z)-T^{(m)}_{\nu_n}(f;Z)\|^{(k)}_p(Q)\leq C\|f\|^{(k)}_p(Q),\ f\in W^{(k)}_p(Q)_Z$.

Let now

$$V_E f = \sum_{\underline{n}\in E} a_{\underline{n}} G^{(m)}_{\underline{n}}(\cdot;Z),$$

then application of theorem 10.19,II, gives for $E=N_\mu=N^d_{2n}(Z)\setminus N^d_n(Z)$ with $n=2^{\mu-1}$

(12.11) $\|V_N f\|_p(Q)\sim\|a\|_{l^\lambda_p}$, $\lambda = \# N_\mu$.

On the other hand by Theorem 9.20,II

(12.12) $\|V_N f\|^{(k)}_p(Q)\sim 2^{k\mu}\|V_N f\|_p(Q)$, $k=0,\dots,m$.

The combination of (12.11) and (12.12) gives

(12.13) $\| V_E f\|^{(k)}_p \leq C \|V_N f\|^{(k)}_p(Q)$ for $E\subset N_\mu$, $k=0,\dots,m$.

In particular for $\nu_n < \nu \leq \nu_{2n}$, $n=2^{\mu-1}$, we find that

$$T^{(m)}_\nu(f;Z) - T^{(m)}_{\nu_n}(f;Z) = V_E f$$

with $E=\varphi^{-1}((\nu_n,\nu>;Z,d)$ and $a_n=(f,G^{(m)}_n(\cdot;Z'))$. Thus, by (12.13) and (12.10) we obtain

$$\|T^{(m)}_\nu(f;Z) - T^{(m)}_{\nu_n}(f;Z)\|^{(k)}_p(Q) \leq C \|f\|^{(k)}_p(Q)$$

what in combination with (12.10) gives finally for $k=0,\dots,m$

(12.14) $\|T^{(m)}_\nu(f;Z)\|^{(k)}_p(Q) \leq C \|f\|^{(k)}_p(Q)$, $f\in W^{(k)}_p(Q)_Z$,

and this completes the proof.

13. Unconditionality in stadard spaces. For given boundary set $Z\subset\partial Q$, $Q=I^d$, as given in (2.37),I, the system $(F^{(m)}_{\underline{n}}(\cdot;Z),\underline{N} \geq \underline{n}(Z))$ is an unconditional basis in each $W^k_p(Q)_Z$, $k=0,\dots,m$, $1<p<\infty$ (see Theorem 9.17,II). Our goal is to establish the same for $(G^{(m)}_{\underline{n}}(\cdot;Z)$, $\underline{n} \geq \underline{n}(Z))$. The idea of the proof is simple. In the first stage , using the technic of singular integrals like in the one-dimensional case in [1], we are going to establish the unconditionality of $(G^{(m)}_{\underline{n}}(\cdot;Z),\ \underline{n} \geq \underline{n}(Z))$ in $W^o_p(Q)_Z=L_p(Q)$. In the next stage we will apply the lifting theorem as established in [2] for the systems $(F^{(m)}_{\underline{n}}(\cdot;Z),\ \underline{n} \geq \underline{n}(Z))$.

We start our discussion with the maximal functions. The maximal function for F Lebesgue measurable on R^d is defined as follows

$$M_dF(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |F(y)|\,dy,$$

where $|E|$ is the Lebesgue measure of E and $B(x,r)=\{y\in R^d: \|x-y\|<r\}$, $\| \ \|$ being the Euclidean norm.

The Hardy-Littlewood-Wiener (cf.[5])theorem sais that M_d is of weak type (1,1) i.e. for some $C<\infty$ and for $F\in L_1(R^d)$ we have

$$(13.1) \qquad |\{x\in R^d: M_dF(x)>\lambda\}| \le \frac{C}{\lambda} \|F\|_{L_1(R^d)}, \lambda>0.$$

For our purpose we need for $f\in L_1(Q)$ the maximal function

$$Mf(x) = \sup_{G\ni x} \frac{1}{|G|} \int_{Q\cap G} |f(y)|\,dy, \ x \ Q,$$

where the sup is taken over all cubes G with sides parallel to the coordinate hyperplanes and such that $x\in G$. It follows directly from (13.1) that for $f\in L_1(Q)$

$$(13.2) \qquad |\{x\in Q: Mf(x)>\lambda\}| \le \frac{C}{\lambda} \|f\|_{L_1(Q)}, \lambda>0.$$

For given $\varepsilon=(\varepsilon_{\underline{n}})$, $\varepsilon_{\underline{n}}=\pm1$, and for $(a_{\underline{n}})$ with finitely many $a_{\underline{n}}\neq0$ let

$$f= \sum_{\underline{n} \ge \underline{n}(z)} a_{\underline{n}} G_{\underline{n}}^{(m)}(\cdot;Z), \quad T_\varepsilon f= \sum_{\underline{n} \ge \underline{n}(Z)} \varepsilon_{\underline{n}} a_{\underline{n}} G_{\underline{n}}^{(m)}(\cdot;Z).$$

<u>Lemma 13.3.</u> There is $C=C(m,d)$, $0<C<\infty$, such that for f and ε given as above we have

$$(13.4) \qquad C^{-1}\|f\|_{L_2(Q)} \le \|T_\varepsilon f\|_{L_2(Q)} \le C\,\|f\|_{L_2(Q)}.$$

<u>Proof.</u> Corollary 10.6,II and Theorem 10.19,II imply that $(G_{\underline{n}}(\cdot;Z), \underline{n} \le \underline{n}(Z))$ is a Riesz basis i.e. unconditional basis in $L_2(Q)$, and this is equivalent to (13.4).

To state the next preliminary results we need some more notation. To each $\underline{n}$ we assign particular dyadic cube $(\underline{n})$ as follows. Using notation from Section 10,II, define $(\underline{n})=I_1\times\ldots\times I_d$ for $\underline{n}=(n_1,\ldots,n_d)\in N_{\mu,e}$ with $\mu\ge 2, \emptyset\neq e\subset\{1,\ldots,d\}$ and

$$I_i=\begin{cases} \left\langle \frac{n_i-1}{2^{\mu-1}}, \frac{n_i}{2^{\mu-1}} \right\rangle & \text{for } i\in D\setminus e \\ \left\langle \frac{n_i-2^{\mu-1}-1}{2^{\mu-1}}, \frac{n_i-2^{\mu-1}}{2^{\mu-1}} \right\rangle & \text{for } i\in e. \end{cases}$$

For $\underline{n}\in N_0\cup N_1$ define $(\underline{n})=Q$.

Proposition 13.5. There are constants $C=C(m,d)<\infty$ and $q=q(m,d)$, $0<q<1$, such that

$$|G_{\underline{n}}^{(m)}(x;Z)G_{\underline{n}}^{(m)}(y;Z')| \leq C\ 2^{\mu d}\ q^{2^{\mu}\|x-y\|}$$

holds for $\underline{n}\in N_{\mu}$, $\mu\geq 0$, $x,y\in Q$.

Proof. Use the results of Sections 8-10, II.

Proposition 13.6. There is $C=C(m,d)<\infty$ such that

$$\sum_{(\underline{n})\subset Q_0} |G_{\underline{n}}^{(m)}(x;Z)G_{\underline{n}}^{(m)}(y;Z')| \leq C\ \frac{|Q_0|}{\|x-y\|^{2d}}, \quad x,y\in Q,\ x\neq y, \tag{13.7}$$

holds for any dyadic cube $Q_0\subset Q$ with sides parallel to the coordinate hyperplanes.

Proof. It is convenient to introduce $\mu_0 = -\frac{1}{d}\log_2|Q_0|$. Now, Proposition 13.5 implies that the left hand side of (13.7) does not exceed ($\delta=\|x-y\|$)

$$\sum_{\mu=\mu_0+1}^{\infty}\ \sum_{(\underline{n})\in N_{\mu}} 2^{\mu d}\ q^{2^{\mu}\delta} = C\sum_{\mu=\mu_0+1}^{\infty} 2^{(\mu-\mu_0-1)d}2^{\mu d}\ q^{2^{\mu}\delta}$$

$$= C\ 2^{(\mu_0+1)d}\sum_{\mu=0}^{\infty} 2^{2\mu d}\ q^{2^{\mu}2^{\mu_0+1}\delta}$$

$$\leq C_1\ 2^{(\mu_0+1)d}\sum_{n=1}^{\infty} n^{2d-1}q^{n2^{\mu_0+1}\delta}$$

$$\leq C_2\ 2^{(\mu_0+1)d}\int_0^{\infty} u^{2d-1}q^{u2^{\mu_0+1}\delta}du$$

$$= C_2\ \frac{1}{2^{(\mu_0+1)d}\delta^{2d}}\int_0^{\infty} v^{2d-1}q^{v}dv$$

$$= C_3\ |Q_0|\,\delta^{-2d}.$$

Lemma 13.7. There is finite $C=C(m,d)$ such that

$$|\{x\in Q: |T_{\varepsilon}f(x)|>\lambda\}| \leq \frac{C}{\lambda}\|f\|_{L_1(Q)}, \quad \lambda>0$$

holds for

$$f = \sum_{\underline{n}\geq\underline{n}(Z)} a_{\underline{n}}G_{\underline{n}}(\cdot;Z)$$

with finitely many $a_{\underline{n}}\neq 0$.

Proof. Using (13.2), Lemma 13.3 and Proposition 13.6 this lemma can be proved in similar way as analogous result in the one-dimensional case. For the details we refer to [1].

Proposition 13.8. Let $1<p<\infty$ and let Z be a boundary set in $Q=I^d$ as given in (2.37),I. Then $(G_{\underline{n}}^{(m)}(\cdot;Z),\underline{n}\geq\underline{n}(Z))$ is an unconditional Schauder basis in $W_p^o(Q)_Z=L_p(Q)$.

Proof. For $1<p<2$ use (13.4), Lemma 13.7 and the Marcinkiewicz interpolation theorem. To cover the case $2<p<\infty$ apply duality argument.

The following is proved in [2].

Proposition 13.9. Let $1<p<\infty, 0\leq l<k\leq m$ and let Z be a boundary set as given in (2.37),I. Then the operator

$$T_{k,l}(Z):\ W_p^k(Q)_Z \longrightarrow W_p^l(Q)_Z$$

defined by

$$T_{k,l}(Z)f=\sum_{n=1}^{\infty} n^{\frac{k-l}{d}}(f,F_n^{(m)}(\cdot;Z'))F_n^{(m)}(\cdot;Z)$$

is a linear isomorphism. Here $(F_n^{(m)}(\cdot;Z))_1$ is a natural ordering of $(F_{\underline{n}}^{(m)}(\cdot;Z),\underline{n}\geq\underline{n}(Z))$.

We are now ready to proof the main result of this section.

Theorem 13.10. Let $1<p<\infty$ and $0\leq k\leq m$. Then $(G_{\underline{n}}^{(m)}(\cdot;Z),\underline{n}\geq\underline{n}(Z))$ is an unconditional basis in each standard space $W_p^k(Q)_Z$; Z being a boundary set as given in (2.37),I.

Proof. For convenience we write $g_{\underline{n}}=G_{\underline{n}}^{(m)}(\cdot;Z)$, $f_{\underline{n}}=F_{\underline{n}}^{(m)}(\cdot;Z)$, $|\ \ |_p^{(k)}=\|\ \ \|_p^{(k)}$ and $(g_n),(f_n)$ for natural orderings of $(g_{\underline{n}}),(f_{\underline{n}})$, respectively. Let $f\in W_p^{(k)}(Q)_Z$ and let

$$f=\sum_{n=1}^{\infty}a_ng_n=\sum_{n=1}^{\infty}b_nf_n\ ,\ f_\varepsilon=\sum_{n=1}^{\infty}\varepsilon_na_ng_n \text{ with } \varepsilon_n=\pm 1.$$

We know by Theorem 9.17,II, that (f_n) is an unconditional basis in each space $W_p^k(Q)_Z$, $k=0,\dots,m$. This and Propositions 13.8, 13.9 imply

$$|f|_p^{(k)}\sim\Big|\sum_{\mu=0}^{\infty}2^{\mu k}\sum_{\underline{n}\in N_\mu}b_{\underline{n}}f_{\underline{n}}\Big|_p^{(o)}=\Big|\sum_{\mu=0}^{\infty}2^{\mu k}\sum_{\underline{n}\in N_\mu}a_{\underline{n}}g_{\underline{n}}\Big|_p^{(o)}$$

$$\sim\Big|\sum_{\mu=0}^{\infty}2^{\mu k}\sum_{\underline{n}\in N_\mu}\varepsilon_{\underline{n}}a_{\underline{n}}g_{\underline{n}}\Big|_p^{(o)}=\Big|\sum_{\mu=0}^{\infty}2^{\mu k}\sum_{\underline{n}\in N_\mu}b'_{\underline{n}}f_{\underline{n}}\Big|_p^{(o)}$$

$$\sim\Big|\sum_{\mu=0}^{\infty}\sum_{\underline{n}\in N_\mu}b'_{\underline{n}}f_{\underline{n}}\Big|_p^{(k)}=\Big|\sum_{\mu=0}^{\infty}\sum_{\underline{n}\in N_\mu}\varepsilon_{\underline{n}}a_{\underline{n}}f_{\underline{n}}\Big|_p^{(k)}=|f_\varepsilon|_p^{(k)},$$

and this completes the proof.

Theorem 13.11. Let $1<p<\infty$ and let

$$R_{k,l}(Z)f=\sum_{n=1}^{\infty} n^{\frac{k-l}{d}} (f,G_n^{(m)}(\cdot;Z'))G_n^{(m)}(\cdot;Z),$$

where the ordering is natural. Then for $0\le l<k\le m$ and Z given by (2.37),I,

$$R_{k,l}(Z):\ W_p^k(Q)_Z \longrightarrow W_p^l(Q)_Z$$

is a linear isomorphism.

Proof. We use the notation introduced in the previous proof. The unconditionality of both bases (f_n) and (g_n), and Proposition 13.9 give for $f\in W_p^k(Q)_Z$

$$|R_{k,l}(Z)f|_p^{(l)} \sim \Big|\sum_{\mu=0}^{\infty} 2^{\mu\frac{k-l}{d}} \sum_{\underline{n}\in N_\mu} a_{\underline{n}} g_{\underline{n}}\Big|_p^{(l)}$$

$$= \Big|\sum_{\mu=0}^{\infty} 2^{\mu\frac{k-l}{d}} \sum_{\underline{n}\in N_\mu} b_{\underline{n}} g_{\underline{n}}\Big|_p^{(l)} \sim \Big|\sum_{n=1}^{\infty} n^{\frac{k-l}{d}} b_n f_n\Big|_p^{(l)}$$

$$\Big|\sum_{n=1}^{\infty} b_n f_n\Big|_p^{(k)} = |f|_p^{(k)} .$$

14. The proof of Theorems C and $\mathring{C}$. The construction of (φ_n) in Theorem C is the same as in Section 11,II. The only difference is that now we use in the standard spaces $W_p^k(Q)_Z$ the bases $(G_n^{(m)}(\cdot;Z))$ in some natural ordering (independent of k). This way (C.1) and ($\mathring{C}$.1) follow from Theorem 12.8. The statements (C.2) and ($\mathring{C}$.2)are a consequence of Theorem 13.10. Finally, (C.4) and ($\mathring{C}$.4) follow by Theorem 3.11.

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