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On Calderoñ-Mitjagin couples of Banach lattices.

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The purpose of this note is to present some new results concerning real interpolation spaces. In particular we are interested in the case when all interpolation spaces with respect to a couple of Banach spaces are described by real interpolation methods.

Let $\bar{X}=(X_0,X_1)$ be a compatible couple of Banach spaces and put, for $x \in X_0+X_1$, $t>0$

$$K(t,x,\bar{X}) = K(t,x) = \inf\left\{ \|x_0\|_{X_0} + t\,\|x_1\|_{X_1} : x=x_0+x_1 \right\}$$

(K-functional). Denote by L^∞ respectively $L^\infty_{1/t}$ the space of all measurable functions x on R_+ such that $|x(t)|$ respectively $|x(t)|/t$ is essentially bounded. Put $\bar{L}^\infty=(L^\infty, L^\infty_{1/t})$. For any exact interpolation space E with respect to $\bar{L}^\infty$, the real interpolation space (or K space) $\bar{X}_{E:K}$ is defined to consist of all $x \in X_0+X_1$ such that $K(\cdot,x,\bar{X}) \in E$. We denote by $I_K(\bar{X})$ respectively $I(\bar{X})$ the collection of all K spaces respectively all exact interpolation spaces with respect to $\bar{X}$. Observe that in particular if $E=L^p_{t^{-\theta}}(R_+,dt/t)$, $0<\theta<1$, $0<p\le\infty$, the space $\bar{X}_{E:K}$ coincides with the familiar space $\bar{X}_{\theta,p}$ of Lions-Peetre.

Mainly due to the efforts of Brudnyǐ-Krugljak we now have a good calculus for the family I_K of K spaces as well as for the analogous family I_J of J spaces. In particular these families are closed for reiteration, duality etc. The main achievement of Brudnyǐ-Krugljak is however their result about K-divisibility which reads as follows:

Theorem I. Let $x \in X_0+X_1$ and let $\varphi_i(t)$, $i\geq 0$ be a sequence of positive concave functions such that $\sum_{i\geq 0}\varphi_i(1) < \infty$ and for all $t>0$ holds

$$K(t,x) \le \sum_{i\geq 0} \varphi_i(t).$$

Then there exists a constant γ, depending only on $\bar{X}$, and elements $x_i \in X_0+X_1$ such that $x=\sum_{i\geq 0} x_i$ (convergence in X_0+X_1) and such that for all $t>0$ and all $i\geq 0$ holds

$$K(t,x_i) \le \gamma\, \varphi_i(t).$$

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For the constant γ one has the universal bound $\gamma \leq (\sqrt{2}+1)^2$ due tc A.A.Dmitriev. The importance of this theorem may be realized by noting that many of the nontrivial results concerning J and K spaces (e.g. reiteration, the sharp form of the fundamental lemma, the equivalence theorem,..) are in fact equivalent to Theorem I. For a proof of this theorem we refer to either [3], [4], or [8]. For a further discussion of J and K spaces one may consult the review article [8] by Ovčinnikov

As an example of what may be achieved by using Theorem I let us state

Theorem 2. Let $0<\theta_0<\theta_1<1$, $1\leq p_0,p_1<\infty$. Then

$$I_K(\bar{X}_{\theta_0 p_0},\bar{X}_{\theta_1 p_1}) = I_K(X_0,\bar{X}_{\theta_1 p_1}) \cap I_K(\bar{X}_{\theta_0 p_0},X_1).$$

The proof of this fact is mainly a combination of the techniques of [1] and Theorem 1. Details will appear elsewhere.

Clearly $I_K(\bar{X}) \subset I(\bar{X})$ and it is of particular interest to know when these two classes coincide i.e. when every space in $I(\bar{X})$ is (possibly after equivalent renorming) also in $I_K(\bar{X})$. If this is the case we say that $\bar{X}$ is a Calderón-Mitjagin couple. It turns out that this is equivalent to the following condition: there exists a constant $\lambda<\infty$ such that whenever $K(t,y) \leq K(t,x)$ holds for all $t>0$ then there exists a linear operator $T\in \mathcal{L}(X_0)\cap\mathcal{L}(X_1)$ with norms less than λ such that $Tx = y$.

The most important examples of Calderón-Mitjagin couples are (L^{p_0},L^{p_1}), $1\leq p_0,p_1\leq\infty$ and $(H^1(T), H^\infty(T))$. The last one, due to P.Jones, is of some interest since in some sense it is the only known "honest" example of a Calderón-Mitjagin couple which is not a couple of Banach lattices. For further examples as well as specific references see [6]. In this connection let us also mention the following

Conjecture. $\bar{X}$ is a Calderón-Mitjagin couple if and only if for all $0<\theta<1$ the complex interpolation spaces $[\bar{X}]_\theta$ are K spaces.

Now let (Ω,Σ,μ) be a separable measure space and let $\bar{X}=(X_0,X_1)$ be a couple of order continuous Köthe function spaces on (Ω,Σ,μ) with the Fatou property. For any couple $\bar{w}=(w_0,w_1)$ (weights) of a.e. positive functions we put $\bar{X}_{\bar{w}}=(X_{0,w_0},X_{1,w_1})$. Here X_{i,w_i}, $i=0,1$ consists of all functions x such that $xw_i\in X_i$. Put also $K_{\bar{w}}(t,x)=K(t,x,\bar{X}_{\bar{w}})$. One then has the following result:

Theorem 3. $\bar{X}_{\bar{w}}$ is a Calderón-Mitjagin couple for all couples $\bar{w}$ of weights if and only if each of the spaces X_0 and X_1 is a weighted L^p space on (Ω,Σ,μ) for some value of p, $1\leq p<\infty$.

This theorem will be the consequence of a series of results of a more technical nature, which we shall now describe.

For $0 < \theta < 1$ we define the space $\bar{X}(\theta) = X_0^{1-\theta} X_1^{\theta}$ to consist of all $x \in L^0$ such that $|x| \leq \lambda |x_0|^{1-\theta} |x_1|^{\theta}$ for some λ, $x_i \in B_{X_i}$. This space is normed by $\|x\|_\theta = \inf \lambda$. We say that the couple $\bar{X}$ is of type K_θ if, whenever $x \in \bar{X}(\theta)$, $y \in L^0$, $w_0^{1-\theta} w_1^{\theta} = 1$ and the inequality $K_{\bar{w}}(t,y) \leq K_{\bar{w}}(t,x)$ holds for all $t > 0$, then it follows that $y \in \bar{X}(\theta)$.

Theorem 3. i) $\bar{X}$ is of type K_θ if and only if $\bar{X}(\theta)$ equals a weighted L^p space for some $1 \leq p < \infty$.

ii) Let $0 < \theta_0 < \theta_1 < 1$. Then $\bar{X}$ is of type K_{θ_0} and K_{θ_1} if and only if both X_0 and X_1 equal weighted L^p spaces $1 \leq p < \infty$.

Let us now briefly sketch the proof of the only if part of Theorem 3. Split Ω into a disjoint union of two sets Ω_0 and Ω_1 such that the couple $\bar{X}$ restricted to Ω_0 as well as to Ω_1 consists of couples of infinite dimensional Banach lattices. Put $\bar{X}=\bar{X}|_{\Omega_0}$ and $\bar{Y}=\bar{X}|_{\Omega_1}$. One then readily sees that $\bar{X}$ and $\bar{Y}$ satisfy the following condition: whenever $w_0^{1-\theta} w_1^{\theta} = 1$, $v_0^{1-\theta} v_1^{\theta} = 1$, $x \in \bar{X}(\theta)$, $y \in L^0$ and $K_{\bar{v}}(t,y) \leq K_{\bar{w}}(t,x)$ holds for all $t > 0$ then $y \in \bar{Y}(\theta)$. Similarly $\bar{Y}$ and $\bar{X}$ satisfy an analogous condition.

Let us say that two Banach lattices are relatively decomposable if there exists a constant $M < \infty$ such that whenever one has two sequences $x_1, \ldots, x_n \in X, y_1, \ldots, y_n \in Y$ of pairwise disjoint elements with $\|y_i\|_Y \leq \|x_i\|_X$, $i=1,\ldots,n$ it follows that

$$\Big\| \sum_0^n y_i \Big\|_Y \leq M \Big\| \sum_0^n x_i \Big\|_X .$$

Our main concern will be to show that $\bar{X}(\theta)$ and $\bar{Y}(\theta)$ as well as $\bar{Y}(\theta)$ and $\bar{X}(\theta)$ are relatively decomposable. It then follows that both $\bar{X}(\theta)$ and $\bar{Y}(\theta)$ are weighted L^p spaces for the same value of p in view of a celebrated result due, in various forms, to Bohnenblust, Kakutani, Tzafriri and others (see [7] Theorem 1.b.12, cf. also [5] Proposition 1.4).

First one notices that there exists a constant $\lambda_{\bar{w},\bar{v}}$ such that whenever $x \in \bar{X}(\theta)$, $y \in L^0$, $K_{\bar{v}}(t,y) \leq K_{\bar{w}}(t,x)$ it follows that $\|y\|_\theta \leq \lambda_{\bar{w},\bar{v}} \|x\|_\theta$. We now claim that further holds

$$\lambda = \sup \left\{ \lambda_{\bar{w},\bar{v}} : w_0^{1-\theta} w_1^{\theta} = 1,\ v_0^{1-\theta} v_1^{\theta} = 1 \right\} < \infty .$$

In order to see this take any two subsets $F \subseteq \Omega_0$, $G \subseteq \Omega_1$ and define similarly $\lambda_{\bar{w},\bar{v}}(F,G)$ and $\lambda(F,G)$. The only difference is that we now only consider elements x and y such that $\operatorname{supp} x \subseteq F$ and $\operatorname{supp} y \subseteq G$. We shall need the following "localization" result which is proved using Theorem 1.

Lemma 4. Let $\bigcup_0^N F_i = \Omega_0$, $\bigcup_0^M G_j = \Omega_1$ be disjoint decompositions of Ω_0 and Ω_1. If $\lambda(F_i, G_j) < \infty$ for all $0 \leq i \leq N$, $0 \leq j \leq M$ then $\lambda = \lambda(\Omega_0, \Omega_1) < \infty$.

Let us assume for simplicity that the underlying measure space (Ω,Σ,μ) is nonatomic. We shall also need the following fact:

Theorem 5. Let X_0,X_1 and X be order continuous Banach lattices over a non atomic measure space. i) Let $x\in X_0+X_1$. Then there exist $x_0,x_1 \in X_0+X_1$, disjointly supported such that $x=x_0+x_1$ and, for all $t>0$ and $i=0,1$,

$$K(t,x) \leq 4352\ K(t,x_i).$$

ii) Let $x\in X$. Then there exist $x_0,x_1 \in X$ with disjoint supports such that $x=x_0+x_1$ and for $i=0,1$

$$\|x\|_X \leq 2\ \|x_i\|_X .$$

Now, in order to prove the above claim that $\lambda<\infty$, let us suppose the contrary, that $\lambda=\infty$. Then, for any $C>0$ we can find couples of weights, $\bar{w},\bar{v}$, normalized as above, and elements $x\in\bar{X}(\theta)$ and $y\in\bar{Y}(\theta)$ such that $K_{\bar{v}}(t,y)\leq K_{\bar{w}}(t,x)$, $\|x\|_\theta\leq 1$ and $\|y\|_\theta > C$. Now split both x and y as in Theorem 5 into disjoint sums $x=x_0+x_1$, $y=y_0+y_1$. Put $F_i=\operatorname{supp} x_i$, $G_i=\operatorname{supp} y_i$. By Lemma 4 we may assume that $\lambda(F_1,G_1)=\infty$. If we now apply the argument inductively with $C=2^n$, $n=1,2,\ldots$ we may easily construct normalized couples of weights $\bar{w}^0,\bar{v}^0$ and elements $\tilde{x}\in\bar{X}(\theta)$, $\tilde{y}\in L^0$ such that $\|\tilde{x}\|_\theta \leq 1$, $K_{\bar{v}^0}(t,\tilde{y})\leq K_{\bar{w}^0}(t,\tilde{x})$ for all $t>0$, but $\tilde{y}\notin\bar{Y}(\theta)$. This contradicts the fact that $\lambda_{\bar{w}^0,\bar{v}^0}<\infty$.

The next step is to invoke the following result:

Lemma 6. Let $x_1,\ldots,x_n\in\bar{X}(\theta)$, $y_1,\ldots,y_n\in\bar{Y}(\theta)$ be two sequences of pairwise disjoint elements with $\|y_i\|_\theta\leq\|x_i\|_\theta$, $i=1,\ldots,n$. Then there exist two couples $\vec{w},\vec{v}$ of weights such that $w_0^{1-\theta}w_1^{\theta}=1$, $v_0^{1-\theta}v_1^{\theta}=1$ and, for all $t>0$,

$$K_{\bar{v}}(t,\sum_1^n y_i)\ \leq\ c\ K_{\bar{w}}(t,\sum_1^n x_i).$$

The proof of a special case of this lemma can be found in [5]. It now follows that $\bar{X}(\theta)$ and $\bar{Y}(\theta)$ are relatively decomposable with constant less than $c\lambda$ which completes the proof of one part of Theorem 3´.

Let us now see how the only if part of Theorem 3´ii) follows from the above. By part i) we know that there exist $p_0,p_1\in[1,\infty)$ and weights v_0,v_1 such that $\bar{X}(\theta_i)=L^{p_i}_{v_i}$,$i=0,1$. Define q_0,q_1,w_0,w_1 by $1/p_i=1-\theta_i/q_0+\theta_i/q_1$, $v_i=w_0^{1-\theta_i}w_1^{\theta_i}$, $i=0,1$. The following result applied to the couple $(X_0,\bar{X}(\theta_1))$ will now show that $X_0=L^{q_0}_{w_0}$. Similarly $X_1=L^{q_1}_{w_1}$.

Theorem 7. Let $0<\theta<1$ and let $\bar{X}$ be a couple of Köthe function spaces with the Fatou property. Then $x\in X_0$ if and only if $|x|^{1-\theta}|x_1|^{\theta}\in\bar{X}(\theta)$

for all $x_1 \in X_1$. Furthermore

$$\|x\|_{X_0} = \sup \left\{ \left\| |x|^{1-\theta} |x_1|^{\theta} \right\|_{\theta}^{1/1-\theta} : x_1 \in B_{X_1} \right\} .$$

For a proof of Theorem 7 as well as a more detailed proof of Theorem 3' see [6]. See also [5] for some related results.

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