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ON THE MULTIVARIATE EULER-FROBENIUS POLYNOMIALS

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The purpose of this note is to generalize certain formulas which are important in the context of univariate cardinal spline interpolation. Specifically, we will extend into a multivariate setting some formulas proved by I.J. Schoenberg in [2] for the exponential Euler polynomial and Euler-Frobenius polynomial. For this purpose we recall [1] that for any set $X = \{x^1,\ldots,x^n\}$ of vectors in $\mathbb{R}^s \setminus \{0\}$ the box spline $B(x|X)$ is defined by requiring that the relation

$$(1) \qquad \int_{\mathbb{R}^s} f(x)B(x|X)dx = \int_{[0,1]^n} f(t_1x^1+\ldots+t_nx^n)dt_1\ldots dt_n$$

holds for every continuous function f on $\mathbb{R}^s$.

<u>Definition 1.</u> Given any $X = \{x^1,\ldots,x^n\} \subset \mathbb{Z}^s \setminus \{0\}$ the exponential Euler spline is defined by

$$A(x;z|X) = \Big(\prod_{j=1}^{n} (1-z^{-x^j})\Big)^{-1} \sum_{\alpha\in\mathbb{Z}^s} z^\alpha B(x-\alpha|X) \quad .$$

Here we use the standard notation $z^x = z_1^{x_1},\ldots,z_s^{x_s}$, $x = (x_1,\ldots,x_s)$.

In the univariate case, $s = 1$, when $X_n = \{\underbrace{1,\ldots,1}_{n}\}$ then $(n-1)!A(x;z|X_n)$ equals the exponential Euler-polynomial $A_{n-1}(x;z)$ of degree $n-1$ for the base z defined by I.J. Schoenberg in [2]. The $A_{n-1}(x;z)$ form an Appell sequence of polynomials.

Our first observation extends this to the multivariate case.

<u>Proposition 1.</u> For $Y \subset X$ such that $\langle X\setminus Y\rangle = \mathrm{span}(X\setminus Y) = \mathbb{R}^s$ we have

$$D_Y A(x;z|X) = A(x;z|X\setminus Y)$$

where

$$D_Y = \prod_{y\in Y} D_y \quad \text{and} \quad D_y f = \sum_{j=1}^{s} y_j \frac{\partial f}{\partial x_j} \quad , \quad y = (y_1,\dots,y_s) \quad .$$

Proof. Setting $\Delta_y f(\cdot) = f(\cdot+y) - f(\cdot)$, $\nabla_y f(\cdot) = f(\cdot) - f(\cdot-y)$ and $\Delta_Y f = \Delta_y(\Delta_{Y\setminus\{y\}} f)$, $\nabla_Y f = \nabla_y(\nabla_{Y\setminus\{y\}} f)$ it is easy to see from (1) that

$$(2) \qquad D_Y B(x|X) = \nabla_Y B(x|X\setminus Y)$$

c.f. [1]. Thus by the above definition we obtain

$$D_Y A(x;z|X) = \left(\prod_{j=1}^{n} (1-z^{-x^j})\right)^{-1} \sum_{\alpha\in Z^s} z^\alpha \nabla_Y B(x-\alpha|X\setminus Y)$$

which upon summation by parts yields

$$= \left(\prod_{y\in X\setminus Y} (1-z^{-y})\right)^{-1} \sum_{\alpha\in Z^s} z^\alpha B(x-\alpha|X\setminus Y)$$

as claimed.

As a consequence of this observation we state

Corollary 1. Suppose that $y,x^1,\dots,x^\ell \in X$ and

$$y = \sum_{j=1}^{\ell} c_j x^j \quad .$$

Then

$$A(x;z|X\setminus\{y\}) = \sum_{j=1}^{\ell} c_j \, A(x;z|X\setminus\{x^j\}) \quad .$$

As a bivariate example of Corollary 1 we let $e^i = (\delta_{ij})_{j=1}^{2}$,

$$X_{p,q,r} = \{\underbrace{e^1,\dots,e^1}_{p},\underbrace{e^2,\dots,e^2}_{q},\underbrace{e^1+e^2,\dots,e^1+e^2}_{r}\}$$

and set $A_{p,q,r}(x;z) = A(x;z|X_{p,q,r})$. Then we have

$$\frac{\partial}{\partial x_1} A_{p,q,r}(x;z) = A_{p-1,q,r}(x;z) \quad , \quad p \geq 1 \quad ,$$

$$\frac{\partial}{\partial x_2} A_{p,q,r}(x;z) = A_{p,q-1,r}(x;z) \quad , \quad q \geq 1 \quad ,$$

and

$$A_{p,q,r-1}(x;z) = A_{p-1,q,r}(x;z) + A_{p,q-1,r}(x;z) \quad .$$

Next we will introduce a multivariate version of the Euler-Frobenius polynomial. For this purpose, we recall that the multivariate truncated power $T(x|X)$ is defined by

$$\int_{\mathbb{R}^s} f(x)T(x|X)dx = \int_{\mathbb{R}^n_+} f(t_1x^1 + \ldots + t_nx^n)dt_1\ldots dt_n \quad ,$$

$f \in C_0(\mathbb{R}^s)$, where we assume that

$$0 \notin [X] \quad ,$$

$[X]$ being the convex hull of X .

<u>Definition 2.</u> Given any $X \subset Z^s \setminus \{0\}$ as above the Euler-Frobenius polynomial $\Pi(z|X)$ is defined by

$$\Pi(z|X) = \sum_{\alpha \in Z^s} z^\alpha B(\alpha|X)$$

($\Pi(z|X)$ is a polynomial when $X \subset Z^s_+$).

For the case $s = 1$, $X = X_n$ considered above $\Pi(z|X_n)$ equals $z\Pi_{n-1}(z)/(n-1)!$ where $\Pi_{n-1}(z)$ is the univariate Euler-Frobenius polynomial introduced by I.J. Schoenberg in [2].

In general, we have

<u>Proposition 2.</u>

$$\frac{\Pi(z|X)}{\prod_{y \in X}(1-z^y)} = \sum_{\alpha \in Z^s} T(\alpha|X)z^\alpha \quad . \tag{3}$$

<u>Proof.</u> Using the relation (4.6.17) from [1] which should read

$$B(x|X) = \nabla_X T(x|X)$$

we have for $z = (z_1,\ldots,z_s)$ in the polydisc $|z_i| < 1$, $i = 1,\ldots,s$, that

$$\Pi(z|X) = \sum_{\alpha\in Z^s} z^\alpha \nabla_X T(\alpha|X) = (-1)^{|X|} \sum_{\alpha\in Z^s} z^\alpha (\prod_{y\in X} (z^y-1)) T(\alpha|X)$$

which completes the proof.

Note that equation (3) shows that the power series whose coefficients are the truncated power evaluated at lattice points is a rational function.

In the univariate case, we know that for X_n as above

$$T(x|X_n) = x_+^{n-1}/(n-1)!$$

so that (3) reduces to the following formula of [2]

$$\frac{\Pi_{n-1}(t)}{(1-t)^n} = \sum_{j=0}^{\infty} (j+1)^{n-1} t^j \quad .$$

Finally, let us state a multivariate analog for the difference-differential equation satisfied by the univariate Euler-Frobenius polynomial.

<u>Theorem 1.</u> Suppose $X = \{x^1,\ldots,x^n\}$ and $x^i = e^i = (\delta_{ij})_{j=1}^s$, $i = 1,\ldots,s$, and let

$$X_i = X \setminus \{x^i\} \quad , \quad i = 1,2,\ldots,n \quad .$$

Then

$$\Pi(z|X) = \frac{1}{n-s} \{ \sum_{\ell=1}^{s} (1-z_\ell) z_\ell \frac{\partial}{\partial z_\ell} \Pi(z|X_\ell) + \frac{1}{n-s} \sum_{i=s+1}^{n} z^{x^i} \Pi(z|X_i) \} \quad .$$

<u>Proof.</u> We shall make use of the following

<u>Lemma 1.</u> Suppose that $y, x^1,\ldots,x^m \in X$ and

$$y = \sum_{j=1}^{m} c_j x^j \quad .$$

Then

$$(1-z^y)\Pi(z|X\setminus\{y\}) = \sum_{j=1}^{m} c_j(1-z^{x^j})\Pi(z|X_j) \quad .$$

The proof of this Lemma follows from Corollary 1 by noting that for any $\ell = 1,\dots,n$

$$A(0,z^{-1}|X_\ell) = \frac{(1-z^{x^\ell})\Pi(z|X_\ell)}{\prod_{j=1}^{n}(1-z^{x^j})} \quad .$$

Returning to the proof of Theorem 1, let us compute

$$\frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)z_\ell\frac{\partial}{\partial z_\ell}\Pi(z|X_\ell)$$

$$= \frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)z_\ell\frac{\partial}{\partial z_\ell}\Big(\prod_{\substack{i=1\\ i\neq\ell}}^{n}(1-z^{x^i})\Big)\sum_{\alpha\in Z^s}T(\alpha|X_\ell)z^\alpha$$

$$+ \frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)z_\ell\Big(\prod_{i\neq\ell}(1-z^{x^i})\Big)\sum_{\alpha\in Z^s}\alpha_\ell\, T(\alpha|X_\ell)z^{\alpha-e^\ell}$$

$$= \frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)z_\ell\frac{\partial}{\partial z_\ell}\Big(\prod_{\substack{i=1\\ i\neq\ell}}^{n}(1-z^{x^i})\Big)\frac{1}{\prod_{i\neq\ell}(1-z^{x^i})}\Pi(z|X_\ell)$$

$$+ \prod_{i=1}^{n}(1-z^{x^i})\sum_{\alpha\in Z^s}\Big(\frac{1}{n-s}\sum_{\ell=1}^{s}\alpha_\ell\, T(\alpha|X_\ell)\Big)z^\alpha \quad .$$

Recalling that

$$T(x|X) = \frac{1}{n-s}\sum_{j=1}^{n}\lambda_j\, T(x|X_j)$$

whenever $x = \sum_{j=1}^{n}\lambda_j x^j$, c.f. [1], we get

$$\frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)z_\ell\frac{\partial}{\partial z_\ell}\Pi(z|X_\ell)$$

$$- \Pi(z|X) = -\frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)\Big(\sum_{i\neq\ell}\frac{1}{(1-z^{x^i})}z^{x^i}x_\ell^i\Big)\Pi(z|X_\ell)$$

$$= -\frac{1}{n-s}\sum_{\ell=1}^{s}(1-z_\ell)\Big(\sum_{i=1}^{n}\frac{1}{(1-z^{x^i})}z^{x^i}x_\ell^i - \frac{z_\ell}{(1-z_\ell)}\Big)\Pi(z|X_\ell) \quad .$$

Using Lemma 1 with $y = x^i$, $i = 1,\dots,n$ and $m = s$ the right hand side of the above equality reads

$$- \frac{1}{n-s} \sum_{i=1}^{n} z^{x^i} \Pi(z|X_i) + \frac{1}{n-s} \sum_{\ell=1}^{s} z_\ell \Pi(z|X_\ell)$$

$$= - \frac{1}{n-s} \sum_{i=s+1}^{n} z^{x^i} \Pi(z|X_i)$$

thereby completing the proof.

In the univariate case, using the fact that $\Pi(z|X_n) = z\,\Pi_{n-1}(z)/(n-1)!$ Theorem 1 reduces to

$$\Pi_{n+1}(t) = (1+nt)\Pi_n(t) + t(1-t)\Pi_n'(t)$$

which is formula (1.9), Lecture 3 of [2].

In the bivariate case $X = X_{p,q,r}$ Theorem 1 gives for $p,q \geq 1$

$$(p+q+r-2)\Pi_{p,q,r}(z_1,z_2) = (1-z_1)z_1 \frac{\partial}{\partial z_1} \Pi_{p-1,q,r}(z_1,z_2)$$

$$+ (1-z_2)z_2 \frac{\partial}{\partial z_2} \Pi_{p,q-1,r}(z_1,z_2) + (p-1)z_1\Pi_{p-1,q,r}(z_1,z_2)$$

$$+ (q-1)z_2\Pi_{p,q-1,r}(z_1,z_2) + r\, z_1 z_2 \Pi_{p,q,r-1}(z_1,z_2) \quad .$$

For our final observation, let us assume that $Y \doteq \{x^1,\ldots,x^s\} \subset X$ satisfies

$$|\det Y| = 1 \quad .$$

Then we notice that Y takes Z^s onto Z^s. Moreover, recalling the relation

$$B(\alpha|Y^{-1}X) = B(Y\alpha|X)$$

from [1] we obtain

$$(4) \qquad \Pi(z|X) = \Pi(z^Y|Y^{-1}X)$$

where we define for any $s \times s$ matrix B with columns $b^1,\ldots,b^s \in \mathbb{R}^s$

$$z^B = (z^{b^1},\ldots,z^{b^s}) \quad .$$

Therefore if we introduce the notation

$$D_j^Y f(y) = \sum_{i=1}^{s} (Y^{-1})_{\ell j} y_i \frac{\partial f(y)}{\partial y_i} \quad , \quad j = 1,\dots,s \ ,$$

we have

Corollary 2. For any $Y = \{x^1,\dots,x^s\} \subset X$, $|Y| = s$, $|\det Y| = 1$

$$\Pi(z|X) = \frac{1}{n-s} \sum_{i=1}^{s} (1-z^{x^i}) D_i^Y \Pi(z|X_i) + \frac{1}{n-s} \sum_{i=s+1}^{n} z^{x^i} \Pi(z|X_i) \quad .$$

This formula follows from Theorem 1 and (4).

This Corollary gives a way to generate the Euler-Frobenius polynomials.

References

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