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SOME APPLICATIONS OF THE SUBORDINATION PRINCIPLE IN FOURIER ANALYSIS

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1. Introduction. The aim of this talk is to illustrate the usefulness of the principle of subordination in Fourier analysis in so far as it yields a simple approach to various Fourier analytic questions: e.g. given a tempered distribution m, how can one decide if it is the symbol of a bounded operator on L^p, if its associated maximal function is bounded, if its Fourier inverse $F^{-1}[m]$ has certain smoothness properties, etc. We shall prove properties of $F^{-1}[m]$ from analogous properties of appropriate Riesz kernels.

Let us first define the Fourier transformation F on $S(\mathbb{R}^n)$ via

$$F[f](\xi) = f^{\wedge}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi x}dx , \qquad f \in S,$$

where S is the test function space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$. By C we denote generic positive constants. Let L^p, $1\le p\le\infty$, be the standard Lebesgue spaces over $\mathbb{R}^n$ with norm $\|\cdot\|_p$ and $[L^p]^{\wedge}$ be the set of Fourier transforms of L^p-functions. A tempered distribution $m \in S'$ is called a Fourier multiplier of type (p,q) if

$$\|m\|_{M_p^q} = \inf\{C : \|F^{-1}[mf^{\wedge}]\|_q \le C \|f\|_p , \quad f \in S\}$$

is finite. Obviously, M_p^q-multipliers generate bounded operators on L^p and in the following we discuss those operators which can be subordinated to suitable Riesz operators. To make this idea more precise, let us consider the classical quasi-convexity criterion:

Lemma 1. Let m be a continuous function on $[0,\infty)$ which vanishes at infinity and satisfies

$$\int_0^\infty t|m''(t)|\, dt = B_m < \infty.$$

Then the composition of m with the absolute value: $m\circ|\cdot|$ is the Fourier transform of an $L^1(\mathbb{R})$-function and

$$\|F^{-1}[m(|\xi|)]\|_1 \le B_m.$$

For the proof consider

$$(1.1) \qquad g(x) = \int_0^\infty \chi_t(x)\, t\, m''(t)dt ,$$

where $\chi_t(x) = (\sin \frac{xt}{2} / \frac{xt}{2})^2/2\pi$ is the familiar Fejér kernel. Then

$$\|g\|_1 \le \int_0^\infty \|\chi_t\|_1 \; t|m''(t)|\, dt = \int_0^\infty t|m''(t)|dt$$

is finite by hypothesis and, by partial integration,

$$(1.2) \qquad g^\wedge(\xi) = \int_0^\infty (1-|\xi|/t)_+ t\, m''(t)dt$$

$$= \int_{|\xi|}^\infty (t-|\xi|)\, m''(t)dt = m(|\xi|)$$

so that Lemma 1 is established.

Now the Fejér kernel χ_t generates the Fejèr operator F_t via convolution: $F_t f = \chi_t * f$. Since $[L^r]^\wedge \subset M_p^q$, $1/q = 1/p + 1/r - 1$, m clearly generates a bounded operator $\mathcal{B}$ on all $L^p(\mathbb{R})$ when m satisfies the hypotheses of Lemma 1. Thus, by (1.1), we may write

$$(1.3) \qquad \mathcal{B} = \int_0^\infty F_t\, t\, m''(t)dt ,$$

i.e., $\mathcal{B}$ can be represented via an integral mean of the Fejér operator and we call $\mathcal{B}$ subordinated to the Fejèr operator by analogy with the subordination introduced in Bochner [2;Ch.4] with respect to the Abel-Poisson operator (see also [19;p.61,p.132], [20;p.46]). We mention that subordination with respect to Riesz summability operators is always archieved by a (generalized) partial integration.

Here our objective is i) to generalize Lemma 1 to several dimensions and ii) to exhibit other instances where the subordination idea is useful.

To this end we interpret $|\xi|$ as a homogeneous distance function with respect to the dilation matrix $A_t = tI$, I being the identity matrix. Thus introduce a dilation matrix $A_t = t^P$, $t > 0$, where P is a real nxn matrix with eigenvalues α_j, $Re\alpha_j > 0$, ν the trace of P, and call a function $\rho : \mathbb{R}^n \to \mathbb{R}_+$ an A_t-<u>homogeneous distance function</u> if

i) ρ is continuous for all $\xi\in\mathbb{R}^n$,

ii) $\rho(\xi) > 0$ for $\xi\in \mathbb{R}^n_o = \mathbb{R}^n \setminus\{0\}$,

iii) $\rho(A_t\xi) = t\rho(\xi)$, $t > 0$.

There is a standard construction for the existence of such a function in Stein and Wainger [21]. But in concrete situations one may consider many different distance functions with respect to the same dilation matrix; e.g. if $A_t = \mathrm{diag}\{t^{1/2}, t^{1/4}\}$, then $\xi_1^2 + \xi_2^4$, $\xi_1^2 + \xi_1\xi_2^2 + \xi_2^4$, $(|\xi_1|^{1/4} + |\xi_2|^{1/2})^8$, etc., are admissible ρ's. Let us mention that an A_t-homogeneous distance function ρ can be compared with the Euclidean distance in the following sense: If $a = \min_{1\le j\le n} Re\alpha_j$, $A = \max_{1\le j\le n} Re\alpha_j$, then we have for every $\varepsilon > 0$

$$(1.4) \qquad \begin{aligned} c_\varepsilon|\xi|^{1/(a-\varepsilon)} &\le \rho(\xi) \le C_\varepsilon|\xi|^{1/(A+\varepsilon)} \quad, |\xi| \to 0, \\ c'_\varepsilon|\xi|^{1/(A+\varepsilon)} &\le \rho(\xi) \le C'_\varepsilon|\xi|^{1/(a-\varepsilon)} \quad, |\xi| \to \infty. \end{aligned}$$

For this and more information see Besov and Lizorkin [1], Calderon and Torchinsky [3], Stein and Wainger [21], Dappa [6].

Here we are interested in formulae analogous to (1.1) where the Fejér kernel χ_t is substituted by Riesz-kernels

$$(1.5)\qquad r_{\lambda,t}(x) = F^{-1}[(1-\rho(\xi)/t)_+^{\lambda}](x)$$

so that, with $R_{\lambda,t}f = r_{\lambda,t} * f$, formula (1.3) transfers to

$$(1.6)\qquad B = \int_0^{\infty} R_{\lambda,t}\, g(t)dt\ ,$$

i.e., we want to discuss operators whose symbols satisfy, analogously to (1.2)

$$(1.7)\qquad m\circ\rho\,(\xi) = C\int_0^{\infty} (1-\rho(\xi)/s)_+^{\lambda}\, s^{\lambda} m^{(\lambda+1)}(s)ds\ ;$$

the fractional derivative can formally be explained by

$$[m^{(\lambda+1)}]^{\wedge}(\omega) = (-1)^{1+[\lambda]}\,(-i\omega)^{\lambda+1}\, m^{\wedge}\ ,\quad \omega\in\mathbb{R}\ ;$$

for the precise definition see Gasper and Trebels [10]; we call $m\circ\rho$ a quasi-radial function. Special cases of (1.6), (1.7) are used e.g. in [22; pp 278] and [21].

To study operators of type (1.6), (1.7) we need more information on the Riesz-kernel $r_{\lambda} = r_{\lambda,1}$.

2. An L^{∞}-estimate for r_{λ} with applications.

Since

$$\sum\nolimits_{\rho} = \{\xi\in\mathbb{R}^n : \rho(\xi) = 1\}$$

is compact [6], we clearly have $r_{\lambda}\in L^{\infty}$. The following theorem shows that the behavior of the Fourier transform of $d\sigma$, the Lebesgue surface measure on $\sum_{\rho}$, and the smoothness of the distance function ρ imply a certain decay of r_{λ} at infinity.

Theorem 1. *Let* $\lambda\geq 0$, $\rho\in C^N(\mathbb{R}^n_o)$, $N\geq[\lambda]+n+4$, *be an* A_t*-homogeneous distance function and*

$$(2.1)\qquad \int_{\Sigma_{\lambda}} e^{ix\xi}\, d\sigma(\xi) = O(|x|^{-\mu}),\quad |x|\to\infty,$$

where O *does not depend upon* x. *Then, for* $|x|\to\infty$,

$$(2.2)\qquad |r_{\lambda}(x)| \leq C\, d(x)^{-\min\{\tau,\nu+1\}},\ \tau = (a-\varepsilon)(\lambda+1+\mu-\varepsilon),$$

where d *is an* A_t'*-homogeneous distance function,* A_t' *being the adjoint of* A_t, *and* $\varepsilon>0$ *is arbitrarily small.*

For the proof let ϕ be a non-negative smooth function on $\mathbb{R}$ with $\operatorname{supp}\phi\subset[\frac{1}{4},\frac{7}{4}]$, which is equal to 1 on $[\frac{1}{2},\frac{3}{2}]$, and set

$$r_{\lambda}^{\wedge}(\xi) = \phi\circ\rho(\xi)r_{\lambda}^{\wedge}(\xi) + R\circ\rho(\xi),\ R(t) = (1-t)_+^{\lambda}\,(1-\phi(t))\ .$$

A slight modification of a method of Randol [17] shows that, by (2.1), the Fourier inverse of the first term on the right side behaves like $|x|^{-\mathrm{Re}\lambda-1-\mu+\varepsilon}$ for integer $\mathrm{Re}\lambda$; interpolation of analytic families of operators gives the same decay in the fractional case whence the assertion for the first term follows, since by (1.4) there holds $d(x)^{a-\varepsilon} \le C|x|$. Concerning $R\circ\rho$ first choose $\Lambda > 0$ such that $D^\sigma\rho(\xi)^\Lambda$, $|\sigma| \le N$, is continuous at the origin; such a Λ exists since [21]

$$|D^\sigma\rho(\xi)^\Lambda| \le C\,\rho(\xi)^{\Lambda-(A+\varepsilon)|\sigma|}, \quad |\xi| \to 0.$$

Then, by the above cited method of Randol,

$$(2.3) \qquad r_{\lambda,\Lambda,1}(x) = \mathcal{O}(|x|^{\varepsilon-\lambda-\mu-1}), \quad r^{\wedge}_{\lambda,\Lambda,t}(\xi) = (1-\rho(\xi)^\Lambda/t)_+^\lambda .$$

Now observe that by subordination

$$R\circ\rho(\xi) = C\int_0^\infty r^{\wedge}_{\lambda,\Lambda,t}(\xi)t^\lambda \left(\frac{d}{dt}\right)^{\lambda+1} R(t^{1/\Lambda})dt ,$$

and that $F^{-1}[f^\wedge(A_{1/t}\xi)](x) = t^\nu f(A_t' x)$. Thus taking the inverse and using (2.3) we obtain

$$(2.4) \qquad \begin{aligned} |F^{-1}[R\circ\rho](x)| &\le C\{\int_0^{d(x)^{-\Lambda}} t^{\lambda+\nu/\Lambda}|(\frac{d}{dt})^{\lambda+1} R(t^{1/\Lambda})|\,dt \\ &\quad + d(x)^{-\tau}\int_{d(x)^{-\Lambda}}^\infty t^{\lambda+\nu/\Lambda-\tau/\Lambda}|(\frac{d}{dt})^{\lambda+1} R(t^{1/\Lambda})|\,dt\} \\ &\le C\, d(x)^{-\min\{\tau,\nu+1\}} \end{aligned}$$

and Theorem 1 is established.

Remark. The estimate (2.2) is certainly not sharp for most choices of a dilation matrix A_t and corresponding distance functions. Therefore we omit to formulate an analogon of the quasi-convexity criterion as well as a Marcinkiewicz-type multiplier criterion on the basis of Theorem 1 and refer for better versions to Corollaries 3 and 5. There is the open problem to give sharp (anisotropic) estimates for the left side of (2.2) and/or (2.1) (for n = 2 see Randol [16] and Sjölin [18]).

Estimates of type (2.1) were investigated e.g. by Herz, Littman, Randol and Sjölin:

a) [11] If $\sum_\rho$ is convex, $\rho \in C^{[(n+7)/2]}(\mathbb{R}^n_o)$ and the Gaussian curvature does not vanish on $\sum_\rho$, then (2.1) holds with $\mu = (n-1)/2$.

b) [13] If in each $x \in \sum_\rho$, $\rho \in C^{n+3}(\mathbb{R}^n_o)$, k of the principal curvatures are different from zero, then (2.1) is valid with $\mu = k/2$.

c) [18], [6] If $n = 2$, $\rho \in C^k(\mathbb{R}^n_o)$, $k \ge 3$, and if the tangent at each point of $\sum_\rho$ has order of contact less than k, then $\mu = 1/k$.

As an application of (2.2) we sketch how to derive growth estimates for functions of type $F^{-1}[m\circ\rho]$. If we choose λ such that $\nu < (a-\varepsilon)(\lambda+1+\mu-\varepsilon) = \tau \le \nu + 1$, $\varepsilon > 0$

arbitrarily small, then it follows from (2.2) that

$$(2.4) \qquad r_\lambda(x) = O_\varepsilon(d(x)^{-\tau}) \;, \quad \nu < \tau \le \nu + 1 \;.$$

Hence, proceeding as in the proof of Theorem 1, we obtain via the inverse of (1.7)

Corollary 1. Let m be sufficiently smooth, λ be as in (2.4) and ρ be as in Theorem 1. Then

$$|F^{-1}[m\circ\rho](x)| \le C\{ \int_0^{1/d(x)} t^{\lambda+\nu}|m^{(\lambda+1)}(t)|dt$$

$$+ d(x)^{-\tau} \int_{1/d(x)}^{\infty} t^{\lambda+\nu-\tau}|m^{(\lambda+1)}(t)|dt\}.$$

In particular we derive for the generalized Bessel potential kernel

$\hat{G}_{\beta,\rho}(\xi) = (1+\rho(\xi))^{-\beta}$, $\beta > 0$, that

$$G_{\beta,\rho}(x) = \begin{cases} O\,(d(x)^{\beta-\nu}) & , \; \beta < \nu \\ O\,(\log(1+1/d(x))) & , \; \beta = \nu \\ O\,(1) & , \; \beta > \nu \;. \end{cases} \quad , \quad |x| \to 0 \;.$$

Lizorkin [14] obtains the same estimates in the particular case $A_t = \mathrm{diag}(t^{\alpha_1},\dots,t^{\alpha_n})$ $\alpha_j > 0$, for the closely related kernel

$$F^{-1}\;[(\sum_{j=1}^{n} (1+\xi_j^2)^{\alpha_j})^{-\gamma/2}](x) \;.$$

Let us mention that Madych [15] gives an estimate for

$$F^{-1}[h(\xi)e^{-\rho(\xi)}](x) \;,$$

(where $h(A_t\xi) = t^\alpha h(\xi)$, $\alpha > 0$, and $h \in C^\infty(\mathbb{R}^n_o)$ is continuous at the origin) which can also be deduced by the above methods if h is quasi-radial (and not necessarily A_t-homogeneous). Another application of the subordination principle concerns the smoothness behavior of $F^{-1}[m\circ\rho]$. Set

$$\Delta_h f(x) = f(x+h) - f(x),\; \Delta_h^k = \Delta_h\,\Delta_h^{k-1},\quad \delta_t'\, f(x) = f(A_t'x) \;.$$

Corollary 2. Let m be sufficiently smooth, $\lambda > \frac{\nu}{a} - 1 - \mu$ and ρ be as in Theorem 1. Then

$$\|\Delta_h^k\, F^{-1}[m\circ\rho]\|_1 \le C\{\, d(h)^{k(a-\varepsilon)} \int_0^{1/d(h)} t^{\lambda+k(a-\varepsilon)}|m^{(\lambda+1)}(t)|dt$$

$$+ \int_{1/d(h)}^{\infty} t^\lambda|m^{(\lambda+1)}(t)|dt\}.$$

It follows from the inverse of (1.7) that

$$(2.5) \qquad \Delta_h^k\, F^{-1}[m\circ\rho](x) = C\int_0^\infty \Delta_h^k\, r_{\lambda,t}(x)\; t^\lambda\; m^{(\lambda+1)}(t)dt \;.$$

Now

$$\Delta_h^k\, r_{\lambda,t}(x) = \Delta_h^k\, t^\nu r_\lambda(A_t'x) = t^\nu \delta_t'\, \Delta_{A_t'h}^k\, r_\lambda(x)$$

so that

$$\|\Delta_h^k r_{\lambda,t}\|_1 \le C \min\{\|r_\lambda\|_1, |A_t'h|^k \sum_{|\sigma|=k} \|D^\sigma r_\lambda\|_1\}$$

$$\le C \min\{1, d(A_t'h)^{k(a-\varepsilon)}\}. \tag{2.6}$$

(Clearly, by Bernstein's inequality for entire functions of exponential type, $r_\lambda \in L^1$ implies that also all its derivatives are integrable.) Taking norms in (2.5) and using (2.6) leads to the assertion.

In particular we obtain for the generalized Bessel potential kernel

$$\|\Delta_h^k G_{\beta,\rho}\|_1 = O(d(h)^\beta) \quad , \quad \beta < ak ,$$

which contains the standard smoothness property of the classical Bessel kernel $F^{-1}[(1+|\xi|^2)^{-\gamma}]$, when we choose $A_t = \mathrm{diag}(t^{1/2},\dots,t^{1/2}) = A_t'$, $d(h) = h_1^2+\dots+h_n^2$, $\gamma = 2\beta$. Clearly the same method allows to estimate $\|\Delta_h^k G_{\beta,\rho}\|_p$. For a discussion of Corollaries 1 and 2 in the radial case see [23].

3. Further illustrations of the subordination principle.

One can obtain more information on the Riesz kernel via L^2-techniques. A very elementary one is the so-called Carlson- Beurling inequality (cf.[8]).

__Lemma 2.__ _Let $1 \le p < 2$, $\kappa = n(\frac{1}{p} - \frac{1}{2})$ and $N > \kappa$ be an integer; let g be measurable with weak derivatives $D^\sigma g \in L^2$, $|\sigma| \le N$. Then_

$$\|F^{-1}[g]\|_p \le C \|g\|_2^{1-\kappa/N} \sum_{|\sigma|=N} \|D^\sigma g\|_2^{\kappa/N} .$$

We cannot apply Lemma 2 directly to $r_N^\wedge$, since the same difficulty as in the proof of Theorem 1 arises: we do not know if all $D^\sigma\rho$, $|\sigma| = N$, are square integrable at the origin when $\rho \in C^N(\mathbb{R}_o^n)$. But $\rho^\Lambda \in C^N(\mathbb{R}^n)$, if we choose e.g. $\Lambda = 2 + [AN]$. Thus

__Theorem 2.__ _If the integer N is such that $N > n(\frac{1}{p}-\frac{1}{2})$, $\rho \in C^N(\mathbb{R}_o^n)$ is an A_t-homogeneous distance function, and Λ is as above, then_

$$(1-\rho(\xi)^\Lambda)_+^N \in [L^p]^\wedge, \quad 1 \le p < 2.$$

If we proceed as in Lemma 1 it turns out that

$$\|F^{-1}[m \circ \rho^\Lambda]\|_p \le C \int_0^\infty t^{N+\nu/\Lambda p'} |m^{(N+1)}(t)|\, dt.$$

Replacing this time $m(t)$ by $m(t^{1/\Lambda})$ we arrive - after an elementary calculation - at (see [8]; for related results in the case $A_t = \mathrm{diag}(t^\alpha,\dots,t^\alpha)$ see Peetre [25]).

__Corollary 3.__ _Let $\rho \in C^N(\mathbb{R}_o^n)$, let $N > n(\frac{1}{p} - \frac{1}{2})$ be an integer, and m be a sufficiently smooth function on $(0,\infty)$, which vanishes at infinity. Then_

$$\|F^{-1}[m \circ \rho]\|_p \le C \int_0^\infty t^{N+\nu/p'} |m^{(N+1)}(t)|\, dt, \quad 1 \le p < 2.$$

Corollary 3 remains true if N is replaced by (a fractional) $\lambda > n(\frac{1}{p} - \frac{1}{2})$, as is shown in Dappa [7] (where Lemma 2 is sharpened in the sense that the derivatives of order N are replaced by an appropriate hypersingular integral of lower order). Further information on r_λ is contained in results on square functions with respect to Riesz kernels.

Theorem 3. Let $\rho \in C^{[\frac{n}{2}+1]}(\mathbb{R}^n_o)$ and define the square function g_λ by

$$g_\lambda(f;x) = (\int_0^\infty |R_{\lambda,t}(f;x) - R_{\lambda-1,t}(f;x)|^2 \frac{dt}{t})^{1/2} \quad , \quad f \in S.$$

Then

$$|\{x : g_\lambda(f;x) > s\}| \le \frac{C}{s} \|f\|_1 ,$$

$$\|g_\lambda(f)\|_p \le C \|f\|_p \quad , \; \lambda > n|\frac{1}{p} - \frac{1}{2}| + \frac{1}{2} \; , \quad 1 < p < \infty.$$

The proof of Theorem 3 in [9] follows standard pattern (see e.g. Igari and Kuratsubo [12] in the instance $\rho(\xi) = |\xi|^2$) and makes heavily use of

$$\| \,|\cdot|^\beta \{|F^{-1}[\rho(\xi)(1-\rho(\xi))_+^\lambda]| + |\nabla F^{-1}[\rho(\xi)(1-\rho(\xi))_+^\lambda]|\}\|_2 = O(1)$$

provided $\mathrm{Re}\,\lambda + \frac{1}{2} > \beta > \frac{n}{2}$ as is shown in Dappa [6]. Let us mention that in the case $\rho(\xi) = |\xi|$, $A_t = t\,I$, I being the identity matrix, A.Carbery (personal communication) has proved

$$\|g_\lambda(f)\|_p \le C \|f\|_p \quad , \quad 2 \le p < \infty, \quad \lambda > n/2 \; ,$$

and, if $n = 2$

$$\|g_\lambda(f)\|_4 \le C \|f\|_4 \quad , \quad \lambda > 1/2$$

(see [4], [5]).

Corollary 4. Let $\rho \in C^{[\frac{n}{2}+1]}(\mathbb{R}^n_o)$, m be a measurable function on $(0,\infty)$ which vanishes at infinity. Define the maximal operator $M_{m\circ\rho}$ on S by

$$M_{m\circ\rho} f(x) = \sup_{t>o} |F^{-1}[m(\rho(\xi)/t)f^\wedge](x)| \; .$$

If m satisfies

$$\int_0^\infty t^{\lambda-1}|m^{(\lambda)}(t)|dt + (\int_0^\infty |t^\lambda m^{(\lambda)}(t)|^2 \frac{dt}{t})^{1/2} \le B$$

for $\lambda > n|\frac{1}{p} - \frac{1}{2}| + \frac{1}{2}$, then

$$\|M_{m\circ\rho} f\|_p \le C\,B\,\|f\|_p \quad , \quad 1 < p < \infty.$$

A similar result has been obtained independently by Carbery [4]. We reproduce here the proof of [9]. Taking the inverse of (1.7) leads to

$$|F^{-1}[m(\rho(\xi)/t)f^{\wedge}](x)|$$

$$\leq C\{\sum_{j=o}^{k} \int_o^\infty s^\lambda |m^{(\lambda)}(s)|\ |R_{\lambda-1+j,st}(f;x) - R_{\lambda+j,st}(f;x)|\frac{ds}{s}$$

$$+ \int_o^\infty s^{\lambda-1}|m^{(\lambda)}(s)|\ |R_{\lambda+k,st}(f;x)|\ ds\}$$

$$\leq C\ B\{\sum_{j=o}^{k} g_{\lambda+j}(f;x) + \sup_{t>o} |R_{\lambda+k,t}(f;x)|\ \} .$$

By Theorem 3 and a theorem of Zo (see [21;pp.1277],[9]) the right hand side turns out to be bounded in L^p and hence the assertion follows.

Corollary 5. Let m be a bounded, sufficiently smooth function on $(0,\infty)$ and satisfy

$$\|m\|_\infty + \sup_{R>0} (\int_R^{2R} |t^\lambda m^{(\lambda+1)}(t)|^2 \frac{dt}{t})^{1/2} \leq B$$

for $\lambda > n|\frac{1}{p} - \frac{1}{2}| + 1/2$. Then, for $\rho \in C^{[\frac{n}{2}+1]}(\mathbb{R}^n_o)$, there holds

$$\|F^{-1}[m\circ\rho\ f^{\wedge}]\|_p \leq C\ B\ \|f\|_p \quad , \quad f\in S\ , \quad 1<p<\infty.$$

The proof of Theorem 1 of Carbery, Gasper and Trebels [5], where $\rho(\xi) = |\xi|$, $n = 2$, at once carries over to the present situation. Formula (1.7) is used to show

$$\|g(F^{-1}[m\circ\rho\ f^{\wedge}])\|_p \leq C\ B\ \|g_\lambda(f)\|_p ,$$

where

$$g(f;x) = (\int_o^\infty |t^\nu \psi(A'_t\cdot) * f(x)|^2 \frac{dt}{t})^{1/2} ,$$

ψ is a suitable smooth kernel with sompact support disjoint from the origin.

Let us mention that, replacing g_λ by a modification of Stein's g^*_λ-function [19;p.88], Dappa and Luers [24] obtain a better result near $p = 1$ and ∞ : already $\lambda > n/2$ in Corollary 5 is sufficient for $m\circ\rho\in M^p_p$, $1<p<\infty$. There are several reasons to conjecture that the "right" λ-parameter range for $2\leq p<\infty$ in Theorem 3 should be $\lambda>\max\{n(\frac{1}{2} - \frac{1}{p}), \frac{1}{2}\}$ which, by the above, would at once imply better versions of Corollaries 4 and 5.

References

1 O.V. Besov and P.I. Lizorkin. Singular integral operators and sequences of convolutions in L_p spaces. Math.USSR-Sb. 2(1967), 57-76.

2 S. Bochner. Harmonic Analysis and the Theory of Probability. Univ.of California Press, Berkeley, 1955.

3 A.P. Calderon and A. Torchinsky. Parabolic maximal function associated with a distribution. Adv.in Math.16(1975), 1-64.

4 A. Carbery. Radial Fourier multipliers and associated maximal functions. Preprint.

5 A. Carbery, G.Gasper and W.Trebels. Radial Fourier multipliers of $L^p(\mathbb{R}^2)$. Proc.Nat.Acad.Sci. U.S.A. 81(1984). In print.

6 H. Dappa. Quasiradiale Fouriermultiplikatoren. Dissertation, Darmstadt, 1982.

7 H. Dappa. Quasi-radial Riesz multipliers. In preparation.

8 H. Dappa and W. Trebels. On L^1-criteria for quasi-radial Fourier multipliers with applications to some anisotropic function spaces. Analysis Math. 9 (1983), 275-289.

9 H. Dappa and W. Trebels. On maximal functions generated by Fourier multipliers. Preprint.

10 G. Gasper and W.Trebels. A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers. Studia Math.65 (1979), 243-278.

11 C. Herz. Fourier transforms related to convex sets. Ann.Math. 75(1962), 81-92.

12 S. Igari and S. Kuratsubo. A sufficient condition for L^p-multipliers. Pacific J.Math. 38 (1971), 85-88.

13 W. Littman. Fourier-transform of surface carried measures and differentiability of surface averages. Bull.Amer.Math.Soc. 69 (1963), 766-770.

14 P.I. Lizorkin. Nonisotropic Bessel potentials, imbedding theorems for Sobolev spaces $L_p^{(r_1,\dots,r_n)}$ with fractional derivatives. Soviet Math. Dokl. 7(1966), 1222-1226.

15 W.R. Madych. On Littlewood-Paley functions. Studia Math. 50 (1974), 43-63.

16 B. Randol. On the Fourier transform of the indicator function of a planar set. Trans.Amer.Math.Soc. 139 (1969), 271-278.

17 B. Randol. The asymptotic behaviour of a Fourier transform and the localization property for eigenfunction expansions for some partial differential operators. Trans.Amer.Math.Soc.168 (1972), 265-271.

18 P. Sjölin. Multipliers and restrictions of Fourier transforms in the plane. Preprint.

19 E.M. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton Univ.Press.Princeton, 1970.

20 E.M. Stein. Topics in Harmonic Analysis Related to the Littlewood-Paley Theory. Princeton Univ.Press. Princeton, 1970.

21 E.M. Stein and S. Wainger. Problems in harmonic analysis related to curvature. Bull.Amer.Math.Soc. 84 (1978), 1239-1295.

22 E.M. Stein and G. Weiss. Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ.Press, Princeton, 1971.

23 W. Trebels. Estimates for moduli of continuity of functions given by their Fourier transform. In: Lecture Notes in Mathematics 571. Springer, Berlin-Heidelberg-New York, 1977, pp.277-288.

24 H. Dappa and H. Luers. A Hörmander type criterion for quasi-radial Fourier multipliers. Preprint.

25 J. Peetre. Applications de la théorie des espaces d'interpolation dans l'Analyse Harmonique. Ricerche Mat. 15 (1966), 3-36.

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