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ON BIVARIATE CARDINAL INTERPOLATION

C. de Boor, K. Höllig, S.D. Riemenschneider

1. Introduction. With the development of non-tensor product multivariate splines (see the survey [8]), and in particular with the introduction of the "box-splines" on regular grids in the plane [1,2], it was natural to look for a non-tensor product analog of I.J. Schoenberg's beautiful and comprehensive theory of cardinal spline interpolation [10, 12-14]. For a set of vectors $\xi_1,\ldots,\xi_n$ in $\mathbb{Z}^m$, (not necessarily distinct), the box spline M is the distribution defined by the rule

(1.1) $$\langle M,\phi\rangle = \int_{[-1/2,1/2]^n} \phi\Big(\sum_{\nu=1}^{n} \lambda_\nu\xi_\nu\Big)d\lambda, \quad \phi \in C_0^\infty(\mathbb{R}^m).$$

The box spline is a piecewise polynomial of total degree $\le n-m$ with support given by

(1.2) $$\operatorname{supp} M = \{\sum \lambda_\nu\xi_\nu;\ \lambda \in [-1/2,1/2]^n\}.$$

Its Fourier transform,

$$M^\wedge(y) = \prod_{\nu=1}^{n} \operatorname{sinc}(y\cdot\xi_\nu);\ \operatorname{sinc} t = \sin(t/2)/(t/2)$$

indicates strongly that the box spline may be a natural generalization of the univariate cardinal spline.

For a fixed box spline M in $\mathbb{R}^m$, the space of cardinal M-splines is generated by the lattice translates of M,

(1.3) $$S_M := \operatorname{span}\{M(\cdot-j);\ j \in \mathbb{Z}^m\}.$$

The first question to consider is that of "cardinal interpolation". The cardinal interpolation problem for the box spline M is well-posed or correct if and only if for any bounded continuous function f on $\mathbb{R}^m$ there exists an unique bounded spline $I_M f \in S_M$, $I_M f(\cdot) = \sum_{j\in\mathbb{Z}^m} a_j M(\cdot - j)$, which interpolates f at the lattice points $\mathbb{Z}^m$, i.e.

$$I_M f(j) = f(j), \quad j \in \mathbb{Z}^m. \tag{1.4}$$

Once the cardinal interpolation problem has been settled for a suitable class of box splines, it is possible to pursue further the analogy to Schoenberg's theory. We develope a suitable notion of "degree tends to infinity" and study the convergence of the cardinal splines as the degree tends to infinity.

We describe the progress on these two problems in the next sections.

2. Interpolation. An obvious necessary condition for the solvability of the cardinal interpolation problem is the linear independence of the translates of the box splines, i.e. whether the map

$$a_j \to \sum a_j M(\cdot - j), \tag{2.1}$$

is 1-1 on $\mathbb{Z}^m$. It has been shown ([9], [7]) that the mapping (2.1) is 1-1 if and only if $|\det Z| = 1$ for all bases, Z, of $\mathbb{R}^m$ contained in $\{\xi_1,\ldots,\xi_n\}$. For example, in $\mathbb{R}^2$ this means that the only possibilities for cardinal interpolation arise when (up to symmetries)

$$\xi_i \in \{(1,0),(0,1),(1,1)\}, \quad i = 1,\ldots, n. \tag{2.2}$$

Analogous to the univariate case it is not too difficult to show that the cardinal interpolation problem is correct for M if and only if the trigonometric polynomial

$$P_M(x) := \sum_{j\in\mathbb{Z}^m} M(j)e^{ij\cdot x} > 0 \tag{2.3}$$

for all $x \in \mathbb{R}^m$. For $m = 2$ this is possible. Let $n = (n_1,n_2,n_3)$ denote the multiplicities of the directions (1,0),(0,1) and (1,1) respectively, and let M_n be the corresponding box spline. Then we have the following:

Theorem 1 ([4]). For all $n \in \mathbb{Z}_+^3$, (2.3) holds with M_n. Thus, cardinal interpolation is correct for M_n and the cardinal splines S_n it generates.

This indicates that the necessary condition stated above is also sufficient in the case $m = 2$. We conjecture that the necessary condition is always sufficient.

In proving (2.3) in the univariate case, Schoenberg actually found the minimum of the polynomial $P(x)$. The method of proof of Theorem 1 does not yield this, but in the special case $\underline{n} = (n,n,n)$, all the critical points of $P_{\underline{n}}$ can be characterized:

Theorem 2 ([6]). The polynomials $P_{\underline{n}}$, ($\underline{n} = (n,n,n)$), $n \in \mathbb{Z}$), attain their minima at $\pm(\frac{2\pi}{3}, \frac{2\pi}{3}) \bmod 2\pi\mathbb{Z}^2$, their maxima at $(0,0) \bmod 2\pi\mathbb{Z}^2$ and have saddle points at $(\pi,0) \bmod 2\pi\mathbb{Z}^2$ and $(0,\pi) \bmod 2\pi\mathbb{Z}^2$. These are the only critical points of $P_{\underline{n}}$.

The proofs of Theorems 1 and 2 depend heavily on the symmetries of $P_{\underline{n}}$. Let $\mathcal{A}$ denote the group of twelve linear transformations which leave the mesh generated by the three directions (2.2) invariant. This group is generated by the matrices

$$(2.4) \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

which correspond to reflection at the origin and permutations of the directions. The essential symmetry relations are expressed by

$$(2.5) \qquad M_{\sigma(\underline{n})}(Ax) = M_{\underline{n}}(x), \quad A \in \mathcal{A}$$

and

$$(2.6) \qquad P_{\underline{n}}(A^*x+2\pi j) = P_{\sigma(\underline{n})}(x), \quad A \in \mathcal{A}, \ j \in \mathbb{Z}^2 .$$

where $\sigma(\underline{n})$ is the permutation of $\underline{n}$ associated with A and A^* is the transpose of A. It is easily checked that the critical points of Theorem 2 can be found by differentiating the formula (2.6).

3. Convergence of bivariate cardinal interpolation. In the univariate case the following result was established by Schoenberg and others.

Theorem ([13, 14, 10, 11]). (i) If the Fourier transform of f is a tempered distribution with supp $\hat{f} \subset (-\pi,\pi)$, then the (univariate) cardinal spline interpolants $I_m f$ of degree m converge locally uniformly to f on $\mathbb{R}$ as $m \to \infty$. If $\hat{f}$ is a bounded measure, then the convergence is uniform on $\mathbb{R}$. (ii) If a sequence of (univariate) cardinal splines s_m of degree m converge uniformly to a bounded function f on $\mathbb{R}$ as $m \to \infty$, then supp $\hat{f} \subseteq [-\pi,\pi]$.

For a bivariate analog of this theorem we require a suitable definition of "degree tends to infinity". Let $n = (n_1, n_2, n_3)$ be as before and let n' be the second number of n_1, n_2, n_3 written in order of magnitude. We require that the limit

$$\mathbf{N} := \lim_{n' \to \infty} n/n' \text{ exists in } [0,\infty]^3. \tag{3.1}$$

Note that any such limit will have at least one component equal to 1, one component ≥ 1, and one component ≤ 1.

With each possible limit vector $\mathbf{N}$ we associate a set

$$\Omega_{\mathbf{N}} = \{x: \lim_{n' \to \infty} |M_n^{\wedge}(x-j)/M_n^{\wedge}(x)| < 1 , \quad j \in J\} \tag{3.2}$$

where $J = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$. A qualitatively correct picture of $\Omega_{\mathbf{N}}$ is shown in Figure 1. The regions $\Omega_{\mathbf{N}}$ vary continuously with $\mathbf{N}$ in the Hausdorff topology.

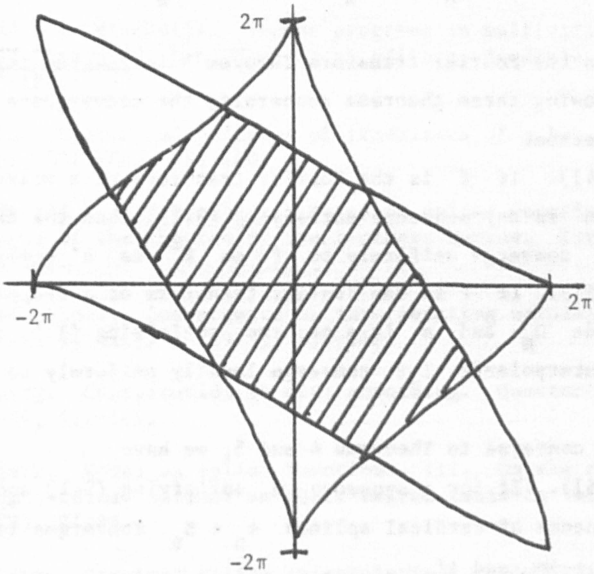

Figure 1

For each limit vector $\mathbf{N}$, the set $\Omega_{\mathbf{N}}$ is a fundamental domain of $\mathbb{R}^2$, i.e. up to a set of measure zero, the translates $2\pi j + \Omega_{\mathbf{N}}$, $j \in \mathbb{Z}^2$, partition the plane. The importance of the sets $\Omega_{\mathbf{N}}$ stems from the fact that their characteristic functions, $\chi_{\mathbf{N}}(x)$, are the limits for the Fourier transforms of the fundamental splines L_n as $n' \to +\infty$. Here $L_n \in S_n$ is the fundamental spline that interpolates the data $\{\delta_j\}_{j \in \mathbb{Z}^2}$. For a point $x \in \mathbb{R}^2$ we set

$$(3.3) \qquad d_{\mathbf{N}}(x) := \mathrm{dist}(x, \partial\Omega_{\mathbf{N}}) .$$

Theorem 3 ([4,5]). (i) Let $\mathbf{n}$ be a sequence satisfying (3.1). Then for sufficiently large n',

$$(3.4) \qquad |L_{\mathbf{n}}^{\wedge}(x) - \chi_{\mathbf{N}}(x)| \le C(1+C'\mathrm{dist}(x,\partial\Omega_{\mathbf{N}})^{-n'}$$

where the constants C, C' do not depend on $\mathbf{n}$, $\mathbf{N}$ or x. ii) If $\mathbf{n}$ also satisfies

$$(3.5) \qquad |\mathbf{n}| := n_1 + n_2 + n_3 \le c(n')^c$$

for some positive constant c, then for any $\varepsilon > 0$ and $\alpha \in \mathbf{Z}_+^2$, there is an n_0' such that for $n' \ge n_0'$ and $d_{\mathbf{N}}(x) > \varepsilon$,

$$|D^{\alpha}(L_{\mathbf{n}}^{\wedge}(x) - \chi_{\mathbf{N}}(x))| \le (1+C'd_{\mathbf{N}}(x))^{-n'} .$$

By passing to the Fourier transform Theorem 3 is the key ingredient for the proof of the following three theorems concerning the convergence of bivariate cardinal interpolation.

Theorem 4 ([4]). If f is the Fourier transform of a measure with support inside $\Omega_{\mathbf{N}}$ and $\mathbf{n}$ is any sequence satisfying (3.1), then the cardinal spline interpolants $I_{\mathbf{n}}f$ converge uniformly to f on $\mathbf{R}^2$ as $n' \to +\infty$.

Theorem 5 ([5]). If f is the Fourier transform of a tempered distribution with support inside $\Omega_{\mathbf{N}}$ and $\mathbf{n}$ is a sequence satisfying (3.1) and (3.5), then the cardinal spline interpolants $I_{\mathbf{n}}f$ converge locally uniformly to f on $\mathbf{R}^2$ as $n' \to +\infty$.

As a partial converse to Theorems 4 and 5, we have

Theorem 6 ([5]). If for a sequence $\mathbf{n}$ satisfying (3.1) and (3.5) some corresponding sequence of cardinal splines $s_{\mathbf{n}} \in S_{\mathbf{n}}$ converges locally uniformly to f on $\mathbf{R}^2$ as $n' \to +\infty$, and if

$$|s_{\mathbf{n}}(x)| \le c(1+|x|)^c, \quad x \in \mathbf{R}^2 ,$$

for all $\mathbf{n}$ and some $c > 0$, then $\mathrm{supp}\,\hat{f} \subseteq \bar{\Omega}_{\mathbf{N}_\alpha}$ for each limit point $\mathbf{N}_\alpha$ of $\mathbf{n}/n'$ in $[0,\infty]^3$.

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Mathematics Research Center
University of Wisconsin
Madison, Wisconsin, U.S.A

Department of Mathematics
University of Alberta
Edmonton, Alberta, Canada