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# NETWORK-NORM ERROR ESTIMATION USING INTERPOLATION OF SPACES AND APPLICATION TO DIFFERENTIAL EQUATIONS

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In order to illustrate the main results of this paper (theorems 1 and 2), the following Cauchy boundary problem for evolutionary equations is considered:

$$(1)\quad \begin{cases} \frac{\partial}{\partial t}u(x,t) - H(\frac{\partial}{\partial x},x,t)u(x,t) = 0 \\ \quad x \quad (-\infty,\infty) = R, \quad 0 < t \leqq T \quad , \\ u(x,0) = f(x),\ x \in R,\ f \in L_p(R)\ (1 \leqq p \leqq \infty). \end{cases}$$

Here $L_p(R)$ is defined as usually. The differential operator H is supposed to allow an unique representation of the solution by a solution operator denoted by G: $u(x,t) = \left[G(t)f\right](x)$.

Problem (1) is being solved numerically by means of problem (2):

$$(2)\quad \begin{cases} u_h(x,t+d) = \sum_{\alpha \in I} c_\alpha u_h(x+\alpha h,t),\ I\text{-finite},\ h \leqq 1 \text{ without l.o.g.}, \\ u_h(x,0) = f(x) \\ x \text{ and } t \text{ belong to the uniform networks } \Sigma_h \text{ and } \Omega_h, \text{ resp.:} \\ x \in \Sigma_h = \left\{x_\mu : x_\mu = \mu h,\ \mu = 0, \pm 1, \pm 2, \dots\right\}, \\ t \in \Omega_h = \left\{t_\nu : t_\nu = \nu d;\ \nu = 0,1,\dots,N,\ \ Nd = T\right\}, \end{cases}$$

$c_\alpha$ and I may depend on x and t. The solution operator of (2) is $G_h(t) : u_h(x,t) = \left[G_h(t)f\right](x)$. The error operator is then $E_h = G_h - G$. All these operators may depend on x, as well as on t.

Many works deal with obtaining estimates of the error $u_h - u$. These are of two types: i) discrete (network) - norm estimates requiring: a) stability, b) approximation and c) f has derivatives of some order (v.[6]); ii) integral $L_p$ - norm error estimates using a) and b) only (these require information about the initial value f a.e. on R instead of $\Sigma_h$ only - v. [1], [2]).

The purpose of the present work is to derive discrete-norm error estimates which, like the integral-norm ones, require conditions a)and b) only, i.e. without the assumption of additional smoothness of f. In order to obtain these estimates, a method is developed (v.also [5]), which is quite general and can be applied to a wide variety of error estimation problems not necessarily related to differential equations. Here applications are restricted to the model evolutionary problem (1) (2) with concrete simple differential operators H only. The technique of estimation includes:

A. Average moduli of smoothness: $\tau_k(f;\delta)_{L_p} = \|\omega_k(f,x;\delta)\|_{L_p}$ where

$$\omega_k(f,x;\ ) = \sup\left\{\left|\Delta_h^k f(t)\right| : t, t+kh \in \left[x-k\delta/_2, x+k\delta/_2\right]\right\}, \text{ where}$$

$$\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m}\binom{k}{m} f(t+mh),\ f \in L_p(R), \text{ bounded, } \delta>0,\ 0<h\leq\delta.$$

(For history, properties and applications of these moduli v. e.g. [4], [7]). Here and henceforth every function $f \in L_p$ is distinguished from its class of equivalence and considered defined by a concrete value in every point of R. The following space is to be dealt with:

$$A_p(R) = \left\{ f : \|f\|_{A_{p,h}} = \|f\|_{L_p} + \tau_1(f;h)_{L_p} < \infty,\ h>0\right\}.$$

For every $h_1, h_2 > 0$ the norms $\|\cdot\|_{A_{p,h_1}}$, $\|\cdot\|_{A_{p,h_2}}$ are equivalent on $A_p$(with constants dependent on $h_1, h_2$).

B. Besov spaces $B^s_{pq}$, interpolation and embedding results.

C. A-spaces (v. [4]), defined by ($1 \leq p,q \leq \infty$, $0<s\leq r$)

$$A^s_{pq}(R) = \left\{ f: \|f\|_{A^s_{pq}} = \|f\|_{L_p} + \left(\int_0^\infty (t^{-s}\tau_r(f;t)_{L_p})^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty\right\}$$

Note that for $\frac{1}{p} < s \leq r$ $A^s_{pq} = B^s_{pq}$ and for $0 < s < \frac{1}{p}$ $A^s_{pq} \subsetneqq B^s_{pq}$.

D. Spaces $M_p$ of Fourier multipliers on $L_p$(v. e.g. [1]).

The discrete-norm spaces to be used are the following:

$$l^p_h(\Sigma_h) = \left\{ f : R \to R,\ \|f\|_{l^p_h(\Sigma_h)} < \infty\right\}, \text{ where}$$

$$\| f \|_{l_h^p(\Sigma_h)} = (\sum_{\mu=-\infty}^{\infty} h \ |f(x_\mu)|^p)^{\frac{1}{p}} \ , \ x_\mu \in \Sigma_h, \ 1 \leq p \leq \infty .$$

$$\tilde{l}_h^p(\Sigma_h) = \{ f : R \to R, \| f \|_{l_h^p(\Sigma_h)} \ , \text{ where}$$

$$\| f \|_{\tilde{l}_h^p(\Sigma_h)} = (\sum_{\mu=-\infty}^{\infty} h \ \sup \{ |f(x)|^p, \ x \in [x_\mu - \tfrac{h}{2}, \ x_\mu + \tfrac{h}{2}]\})^{\frac{1}{p}}.$$

It can be proved that for every $\xi \in R$ $\tilde{l}_h^p(\Sigma_h + \xi) = \tilde{l}_h^p(\Sigma_h) = A_{p,h}(R)$ (equivalence of norms with constants independent of h). With respect to the applications it should be mentioned here that for some important classes of operators, stability in $L_p$ implies stability : $A_p \to l_h^p(\Sigma_h)$. (Such as, for instance, all difference operators $G_h$ appearing in (2) or convolution operators $Ef = \check{\varrho} * f$, with $\varrho$ - Fourier multiplier on $L_p$, $\check{\varrho} = F^{-1}(\varrho)$ - inverse Fourier transform of $\varrho$, and $\varrho \in S$(Schwartz' space) - v.Appl. 1).

The next theorem is an useful generalization of a theorem of V. Popov (v. [3], [4], [7]).

<u>Theorem 1</u>. (v. also [5]). Let E be a Lipschitz operator: $A_p \to l_h^p(\Sigma_h)$ with:

(A) For every $f,g \in A_p$ $\| Ef - Eg \|_{l_h^p(\Sigma_h)} \leq c_1 \| f - g \|_{A_{p,h^{\sigma/r}}}$

(B) For every $f \in W_p^r$(Sobolev space, i.e.$f, f^{(r)} \in L_p$, $f^{(r-1)} \in AC$ - the space of all absolutely continuous functions) it is fulfilled that

$$\| Ef \|_{l_h^p(\Sigma_h)} = c_2 h^{\sigma} \ (\| f \|_{L_p} + \| f^{(r)} \|_{L_p}), \quad r > 0,$$

where $0 < \sigma \leq r$; $c_1$, $c_2$ do not depend on h; $h \leq 1$ (without l.o.g.).

Then, for every $f \in A_p$, $0 < s \leq r$,

$$\| Ef \|_{l_h^p(\Sigma_h)} = c(r, c_1, c_2)(h^{\sigma} \| f \|_{L_p} + \tau_r(f; h^{\frac{\sigma}{r}})_{L_p}) \leq$$

$$\leq c(r, c_1, c_2) h^{\frac{\sigma s}{r}} \| f \|_{A_{p\infty}^s} .$$

When applied to some error operator E (usually linear bounded), this theorem yields a network-norm interpolation estimate. It may happen, however, that the $l_h^p$-norm estimate of condition (B) is not available, <u>while there is an analogous one in $L_p$-norm</u>. In this case the next theorem may prove useful:

<u>Theorem 2.</u> Let E be a Lipschitz operator: $A_p \to l_h^p(\Sigma_h)$, with:

(A) For every $f,g \in A_p$, $\| Ef - Eg \|_{l_h^p(\Sigma_h)} = c_1 \| f-g \|_{A_{p,h^{\mu(r)/r}}}$ ;

(B) For every $f \in W_p^r$, $r \geq 1$,

$$\| Ef \|_{L_p} = c_2 h^{\sigma_r} ( \| f \|_{L_p} + \| f^{(r)} \|_{L_p} ) .$$

(C) For every $f \in W_p^{r-1}$,

$$\| Ef \|_{L_p} \leq c_3 h^{\sigma_{r-1}} ( \| f \|_{L_p} + \| f^{(r-1)} \|_{L_p} ), \text{ where:}$$

$0 < \sigma_r$, $\sigma_{r-1} \leq \sigma_r$ ; $c_1$, $c_2$, $c_3$ do not depend on h; $h \leq 1$ (without l.o.g.).

(D) For every $f \in W_p^1$, $Ef \in AC$ and E commutes with $\frac{\partial}{\partial x}$.

Then, for every $f \in A_p$, $0 < s \leq r$,

$$\| Ef \|_{1_h^p(\Sigma_h)} = c(r, c_1, c_2, c_3) h^{\frac{s\mu(r)}{r}} \| f \|_{A^s_{p\infty}} ,$$

where $\mu(r) = \min \left\{ \sigma_r, \sigma_{r-1} + 1 \right\}$.

Both these theorems hold also for a bounded interval instead of R and can be modified for a non-uniform network as well. They are the main technique (especially Theorem 2) in the following applications:

Application 1. Parabolic equation of arbitrary order (see [1]). For denotations, see (1), (2). In this case $H(\frac{\partial}{\partial x}, x, t) = \frac{\partial^{2m}}{\partial x^{2m}}$, $m > 0$ integer. (We may consider $H(\xi) = \xi^{2m}$).

The operators G and $G_h$ are defined on the network by the convolutions with Fourier multipliers:

$$G(\nu d) f = F^{-1} (\exp(-\nu d H(.)))_* f$$

(3) $\quad G_h(\nu d) f = F^{-1} (\exp(-\nu d H_h(.)))_* f$, where

$$H_h(\xi) = - \frac{\ln e(h\xi)}{d} , \text{ where}$$

$e(h\xi) = \sum_{\alpha \in I} c_\alpha e^{i\alpha h \xi}$ ; $c_\alpha$, I - constants (see (2)) and

$F(G_h(\nu d) f)(\xi) = [e(h\xi)]^\nu \hat{f}(\xi)$ ; $F(f) = \hat{f}$ - Fourier transform of f.

The next theorem is an integral-norm estimate for this problem:

Theorem A. (v. [1]). Let the following be fulfilled:

(A) $G_h(t)$ - stable in $L_p$ (this contains the case of $\sum_{\alpha \in I} |c_\alpha| \leq$

$\leq 1 + cd,\ c > 0$),

(B) $H_h(\xi)$ approximates $H(\xi)$ of the order $r > 0$ (for the definition of this approximation, see [1]).($d = kh^{2m}$).

Then, $\| E_h(t) f \|_{L_p(R)} \leq ch^s \| f \|_{B^s_{p\infty}(R)}$,

$t = \nu d = \nu kh^{2m}$, $= 0, 1, 2, \ldots, N$; $0 < s \leq r$.

Thence, Theorem 2 and some lemmas imply:

<u>Theorem 3.</u> Let the assumptions of theorem A hold, with $f \in A_p$:

Then, $\| E_h(t) f \|_{l^p_h(\Sigma_h)} \leq ch^s \| f \|_{A^s_{p\infty}(R)}$ .

<u>Corollary 1.</u>: If for this problem $f^{(s)} \in L_p(R)$, $1 \leq s \leq r$, then,

$$\| E_h(t) f \|_{l^p_h(\Sigma_h)} = 0(h^s \omega_{r-s}(f^{(s)}; h)_{L_p(R)} + h^r) .$$

Here the integral modulus of smoothness $\omega_k(f;h)_{L_p}$ is defined as usually.

<u>Corollary 2.</u> If $Vf^{(s-1)} < \infty$ (the variation of f) , $1 \leq s \leq r$, then

$$\| Eh(t) f \|_{l^p_h(\Sigma_h)} = 0(h^{s-1+1/p}), \quad 1 \leq p \leq \infty .$$

<u>Application 2.</u> Hyperbolic equation of the first order (v. [2]). Using denotations (1), (2), (3), we have $H(\frac{\partial}{\partial x}, x, t) = \frac{\partial}{\partial x}$, i.e. $H(\xi) = \xi$ . For this problem the following $L_p$-norm estimate holds:

<u>Theorem B.</u> (v. [2]). Let conditions (A) and (B) of Theorem A be fulfilled for $G(t)$ with $H(\xi) = \xi$, $d = kh$, and a corresponding $G_h(t)$.

Then, $\| E_h(t) f \|_{L_p(R)} \leq ch^{\frac{sr}{r+1}} \| f \|_{B^s_{p\infty}(R)}$ , $0 < s < r+1$.

Our corresponding discrete-norm estimate is given by:

<u>Theorem 4</u>. Let the assumptions of theorem B. hold, with $f \in A_p$:

Then, $\| E_h(t) f \|_{l^p_h(\Sigma_h)} = ch^{\frac{sr}{r+1}} \| f \|_{A^s_{p\infty}(R)}$ .

Corollary 3. If $f^{(s)} \in L_p(R)$, $1 \leq s < r+1$, then

$$\| E_h(t)f \|_{l_h^p(\sum_h)} = O(h^{\frac{sr}{r+1}} \omega_{r+1-s}(f^{(s)}; h^{\frac{r}{r+1}})_{L_p(R)} + h^r) .$$

Corollary 4. If $Vf^{(s)} < \infty$, $0 \leq s \leq r$, then

$$\| E_h(t)f \|_{l_h^p(\sum_h)} = O(h^{\frac{(s+1)r}{r+1} - 1 + \frac{1}{p}}), \quad 1 \leq p \leq \infty .$$

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