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ON BIVARIATE HERMITE TRIGONOMETRIC INTERPOLATION

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1. Introduction Methods of bivariate Hermite polynomial interpolation are of considerable importance in computer aided geometric design and in the finite element method [6, 8]. In several papers we have used Boolean methods of multivariate interpolation and approximation [4] to derive explicit formulas for these and new interpolation methods [1, 2]. In this paper we will apply Boolean methods of interpolation to construct a global method of bivariate Hermite interpolation using trigonometric polynomials in two variables.

2. Univariate Hermite trigonometric interpolation In this section we will recall the method of Hermite trigonometric interpolation as developed by Salzer [7]. Let p and q be integers with $p \geq 0$ and $n \geq 1$, and let $\tau_{p,n}$ denote the linear space of trigonometric polynomials given by

$$\tau_{p,n} = \text{lin}\{\cos(kx) : 0 \leq k \leq (p+1)n\} + \text{lin}\{\sin(kx) : 1 \leq k \leq (p+1)n-1\}$$

if p is even, and

$$\tau_{p,n} = \text{lin}\{\cos(kx) : 0 \leq k \leq (p+1)n-1\} + \text{lin}\{\sin(kx) : 1 \leq k \leq (p+1)n\}$$

if p is odd, respectively. Obviously $\tau_{p,n}$ has dimension $2(p+1)n$ in in both cases. Moreover let

$$x_j = j\pi/n \quad , \quad m \in \mathbb{Z} \quad .$$

Let $C^p_{2\pi}(\mathbb{R})$ denote the linear space of 2π-periodic complex-valued functions on $\mathbb{R}$ which have continuous p-th derivatives $D^p f$.

Theorem 1 For any $u \leq p$ there is a unique $H_{u,p} \in \tau_{p,n}$ satisfying

$$D^r H_{u,p}(x_j) = \delta_{0,j}\,\delta_{u,r} \quad , \quad 0 \leq j < 2n \ , \ 0 \leq r \leq p \quad .$$

Given $f \in C^p_{2\pi}(\mathbb{R})$,

$$T_p(f)(x) = \sum_{j=0}^{2n-1} \sum_{r=0}^{p} D^r f(x_j)\, H_{r,p}(x-x_j)$$

is the unique trigonometric polynomial in $\tau_{p,n}$ for which

$$D^r T_p(f)(x_j) = D^r f(x_j) \quad , \quad 0 \le r \le p \;,\quad 0 \le j < 2n \quad .$$

The proof of theorem 1 may be found found in [5] . The special case $p = 1$ was treated by Jackson [3, 9]. Explicit formulas were derived by Kress [5] who also established remainder formulas . We list the fundamental functions $H_{r,p}$ for $p = 0, 1$:

$$H_{0,0}(x) = \sin(nx)\,ctg(x/2)/(2n) \quad ,$$

$$H_{0,1}(x) = (\sin(nx)ctg(x/2)/(2n))^2 ,\; H_{1,1}(x) = \sin(nx)^2 ctg(x/2)/(2n^2).$$

It follows from theorem 1 that T_p is a linear projector on $C^p_{2\pi}(\mathbb{R})$.

<u>Theorem 2</u> The projectors $T_0, T_1, \ldots, T_q$ form a commuting chain on $C^q_{2\pi}(\mathbb{R})$, i. e.,

$$T_p(T_q(f)) = T_q(T_p(f)) = T_p(f) \quad , \quad q \ge p \;,\; f \in C^q_{2\pi}(\mathbb{R}) \quad .$$

<u>Proof:</u> It follows from the definition of the space $\tau_{p,n}$ that $\tau_{p,n}$ is a subspace of $\tau_{q,n}$ for $p \le q$. Since T_q is idempotent we have

$$T_q(f) = f \quad , \quad f \in ran(T_q) = \tau_{q,n} \quad .$$

Because of $T_p(f) \in \tau_{q,n}$ we obtain

$$T_q(T_p(f)) = T_p(f) \quad , \quad p \le q \;,\quad f \in C^q_{2\pi}(\mathbb{R}) \quad .$$

Note that

$$D^u T_q(f)(u) = D^u f(x_j) \quad , \quad 0 \le u \le p \;,\quad 0 \le j < 2n \quad .$$

Now an application of theorem 1 yields

$$T_p(T_q(f)) = T_p(f)$$

which completes the proof of theorem 2.

<u>3. Bivariate Hermite trigonometric interpolation</u>

Let $C_{2\pi}(\mathbb{R}^2)$ denote the linear space of bivariate continuous complex-valued functions on $\mathbb{R}^2$ which are 2π-periodic in both variables , i. e.,

$$f(x+2\pi,y) = f(x,y) \;,\quad f(x,y+2\pi) = f(x,y) \;,\quad x,y \in \mathbb{R} \quad .$$

Given a nonnegative integer m let $C^m_{2\pi}(\mathbb{R}^2)$ denote the linear subspace of $C_{2\pi}(\mathbb{R}^2)$ of functions which have continuous partial derivatives of order

$\leq m$:

$$D^{(j,k)}f = \left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial y}\right)^k f \in C_{2\pi}(\mathbb{R}^2) \quad , \quad 0 \leq j+k \leq m \quad .$$

Let $y_k = x_k$, $k \in \mathbb{Z}$. We define the parametrically extended Hermite interpolation projectors by

$$T_p^1(f)(x,y) = \sum_{j=0}^{2n-1}\sum_{r=0}^{p} D^{(r,0)}f(x_j,y)\, H_{r,p}(x-x_j) \quad ,$$

$$T_q^2(f)(x,y) = \sum_{k=0}^{2n-1}\sum_{s=0}^{q} D^{(0,s)}f(x,y_k)\, H_{s,q}(y-y_k) \quad ,$$

$$p,q \leq m \quad , \quad f \in C_{2\pi}^m(\mathbb{R}^2) \quad .$$

It follows from the lemma of Schwarz that

$$T_p^1 T_q^2 = T_q^2 T_p^1 \quad , \quad p+q \leq m \quad .$$

$T_p^1 T_q^2$ is the projector of tensor product Hermite trigonometric interpolation. Its range satisfies

$$\operatorname{ran}(T_p^1 T_q^2) = \tau_{p,n} \otimes \tau_{q,n} \quad .$$

Moreover we have

$$T_p^1 T_q^2(f)(x,y)$$

$$= \sum_{j=0}^{2n-1}\sum_{k=0}^{2n-1}\sum_{u=0}^{p}\sum_{v=0}^{q} D^{(u,v)}f(x_j,y_k)\, H_{u,p}(x-x_j)\, H_{v,q}(y-y_k)$$

and

$$D^{(r,s)}T_p^1 T_q^2(f)(x_j,y_k) = D^{(r,s)}f(x_j,y_k) \ , \ r \leq p, \ s \leq q, \ \ 0 \leq j,k < 2n$$

Our main objective is to solve the complete Hermite interpolation problem in the space of bivariate trigonometric polynomials defined by

$$\tau_{m,n}^2 = \sum_{p+q=m} \tau_{p,n} \otimes \tau_{q,n} \quad .$$

It follows from theorem 2 and the lemma of Schwarz that the product projectors

$$T_0^1 T_m^2 \ , \ T_1^1 T_{m-1}^2 \ , \ldots, \ T_m^1 T_0^2$$

commute on $C_{2\pi}^m(\mathbb{R}^2)$. Thus, we may apply the method of Boolean sum interpolation as developed in [1, 2,4]. In particular, we introduce the Boolean sum projector

$$B_m = T_0^1 T_m^2 \oplus T_1^1 T_{m-1}^2 \oplus \ldots \oplus T_m^1 T_0^2 \quad .$$

We recall that the Boolean sum of two commutative projectors P and Q is defined by $P \oplus Q = P + Q - PQ$. It follows from theorem 2 that

$$T_p^1 T_q^1 = T_q^1 T_p^1 = T_p^1 \quad , \quad T_p^2 T_q^2 = T_q^2 T_p^2 = T_p^2 \quad , \quad 0 \le p \le q \le m \quad .$$

Thus, B_m is a projector of generalized Biermann interpolation which is explicitly given by

$$B_m = T_0^1 T_m^2 + T_1^1 T_{m-1}^2 + \ldots + T_{m-1}^1 T_1^2 + T_m^1 T_0^2$$
$$- T_0^1 T_{m-1}^2 - T_1^1 T_{m-2}^2 - \ldots - T_{m-1}^1 T_0^2$$

(see [2]) . Applying the general relation

$$\mathrm{ran}(P \oplus Q) = \mathrm{ran}(P) + \mathrm{ran}(Q)$$

to the projector B_m we obtain

$$\mathrm{ran}(B_m) = \tau_{m,n}^2 \quad .$$

<u>Theorem 3</u> Let $f \in C_{2\pi}^m(\mathbb{R}^2)$. Then $g = B_m(f)$ is the unique trigonometric polynomial in the space $\tau_{m,n}^2$ which solves the complete Hermite interpolation problem :

$$D^{(u,v)} g(x_j, y_k) = D^{(u,v)} f(x_j, y_k)$$

$$\text{for} \quad 0 \le u+v \le m \quad \text{and} \quad 0 \le j,k < 2n \quad .$$

<u>Proof:</u> It follows from the rules of Boolean sum interpolation that

$$T_p^1 T_q^2 B_m = T_p^1 T_q^2 \quad , \quad p+q = m \quad .$$

Using the interpolation properties of $T_p^1 T_q^2$ we get for $r \le p$, $s \le q$ and $p+q = m$:

$$D^{(r,s)} g(x_j, y_k)$$

$$= D^{(r,s)} T_p^1 T_q^2 (g)(x_j, y_k)$$

$$= D^{(r,s)} T_p^1 T_q^2 B_m (f)(x_j, y_k)$$

$$= \quad D^{(r,s)} T_p^1 T_q^2 (f)(x_j, y_k)$$

$$= \quad D^{(r,s)} f(x_j, y_k) \quad ,$$

i. e., $g = B_m(f)$ satisfies the complete Hermite interpolation properties. To establish uniqueness of g we use the duality relation of Boolean sum interpolation :

$$I - P \oplus Q \quad = \quad (I - P)(I - Q)$$

where I is the identity operator . Suppose that $h \varepsilon \tau^2_{m,n}$ satisfies

$$D^{(u,v)} h(x_j, y_k) \quad = \quad D^{(u,v)} g(x_j, y_k) \ , \quad u+v \leq m, \quad 0 \leq j,k < 2n \quad .$$

Then we have

$$(I - T_p^1 T_q^2)(g - h) \quad = \quad g - h \quad , \quad p+q = m \ ,$$

which implies

$$(I - B_m)(g - h) \quad = \quad \prod_{p+q=m} (I - T_p^1 T_q^2)(g - h) \quad = \quad g - h \quad .$$

Thus we obtain $B_m(g-h) = 0$ which yields

$$g \quad = \quad B_m(g) \quad = \quad B_m(f) \quad = \quad B_m(h) \quad = \quad h$$

in view of $g,h \ \varepsilon \ \mathrm{ran}(B_m) = \tau^2_{m,n}$. This completes the proof of therem 3 .

As an example we consider the case $m = 1$. Note first that

$$B_1 \quad = \quad T_0^1 T_1^2 \ + \ T_1^1 T_0^2 \ - \ T_0^1 T_0^2 \quad .$$

For any $f \ \varepsilon \ C^1_{2\pi}(\mathbb{R}^2)$ we have

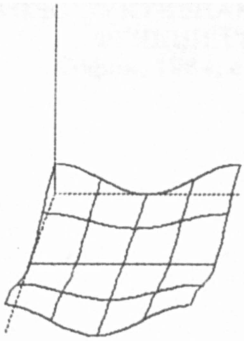
$$B_1(f)(x,y)$$

$$= \quad \sum_{u+v \leq 1} \ \sum_{j=0}^{2n-1} \ \sum_{k=0}^{2n-1} D^{(u,v)} f(x_j, y_k) G^1_{u,v}(x-x_j, y-y_k)$$

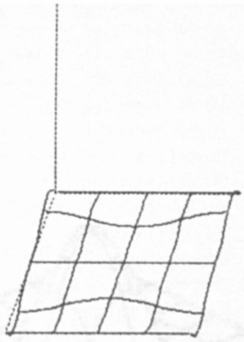
with

$$G^1_{0,0}(x,y) = H_{0,1}(x) H_{0,0}(y) + H_{0,0}(x) H_{0,1}(y) - H_{0,0}(x) H_{0,0}(y) \quad ,$$

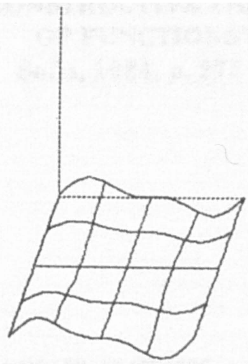
$$G^1_{1,0}(x,y) = H_{1,1}(x) H_{0,0}(y) \ , \ G^1_{0,1}(x,y) = H_{0,0}(x) H_{1,1}(y) \quad .$$



$G^{1}_{0,0}(x,y)$



$G^{1}_{1,0}(x,y)$



$G^{1}_{0,1}(x,y)$

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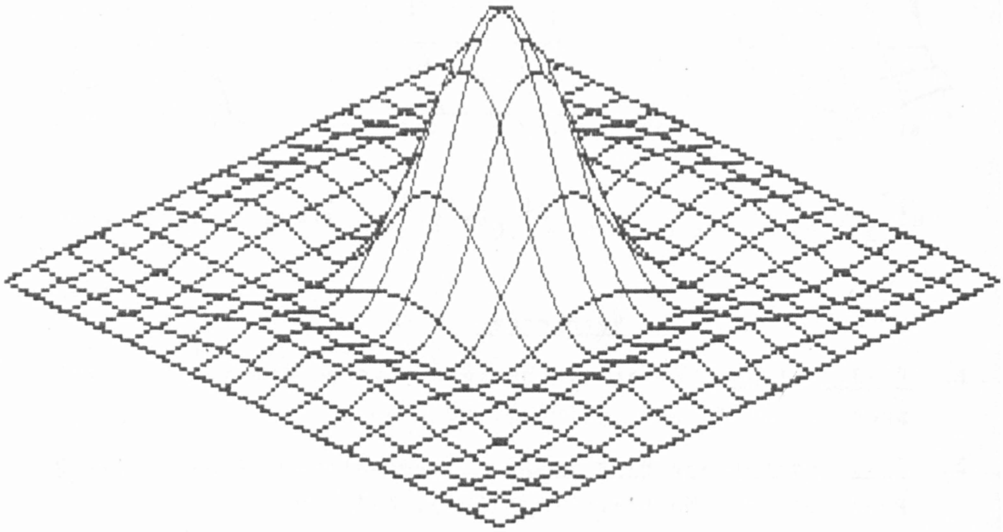

n= 2 , JX= 2 , JY= 2