КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 273—283

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 273-283

SOME NEGATIVE RESULTS ON THE COMPARISON OF APPROXIMATION PROCESSES

W. Dickmeis, R.J. Nessel, and E. van Wickeren

The present note may be considered as a contribution to problems, posed by J. Favard and S.B. Steckin on the occasion of previous conferences on the constructive theory of functions.

Indeed, about twenty years ago, at the first Oberwolfach conference ON APPROXIMATION THEORY, Favard raised the question for the comparison of approximation processes (cf. [13; 14]). In connection with the uniform boundedness principle (UBP) he pointed out that one feature of a solution to this complex problem would certainly consist in deriving significant negative results. Let us recall his contribution in the following terminology.

Let X be a Banach space (with norm $\|\cdot\|$), and X^* be the class of functionals T on X which are sublinear, i.e.,

$$|T(f+g)| \leq |Tf| + |Tg| , \qquad |T(af)| = |a|\ |Tf|$$

for all $f,g \in X$ and scalars a, and which are bounded, i.e.,

$$\|T\|_{X^*} := \sup\{|Tf| : \|f\| \leq 1\} < \infty .$$

Then, given a sequence $\{T_n\} \subset X^*$ (e.g., remainder functionals), the classical UBP may (equivalently) be formulated as a resonance principle (cf. [15, p. 20]).

<u>UBP:</u> *Suppose that for* $T_n \in X^*$ *there are elements* $h_n \in X$ *such that (*$\mathbb{N}$ *:= set of natural numbers)*

$$(1) \qquad \|h_n\| \leq C_1 \qquad (n \in \mathbb{N}) ,$$

$$(2) \qquad |T_n h_n| \neq O(1) \qquad (n \to \infty) .$$

Then there exists a counterexample $f_o \in X$ *such that*

(3) $$|T_n f_o| \neq O(1) \qquad (n \to \infty) .$$

Given two sequences $\{R_n\},\{V_n\} \subset X^*$, the (strong) comparison problem asks for an estimate of type

(4) $$|R_n f| = O_f(|V_n f|) \qquad (n \to \infty)$$

to be valid for each f of a prescribed class in X (see, e.g., (17) for a concrete interpretation). In this connection it was Favard [13; 14] who applied the UBP to the ratio $R_n/\|V_n\|_{X^*}$ in order to deduce the following negative result.

Corollary: Suppose that for $R_n, V_n \in X^*$ *there are elements* $h_n \in X$ *satisfying (1) and* $|R_n h_n| \neq O(\|V_n\|_{X^*})$. *Then there exists a counterexample* $f_o \in X$ *such that*

(5) $$|R_n f_o| \neq O(|V_n f_o|) \qquad (n \to \infty) .$$

The conditions of this first contribution depend upon the quantity $\|V_n\|_{X^*}$, i.e., upon the supremum of the values of $|V_n f|$ over the unit ball in X. From the point of view of applications, this may be too large. In fact, it would be desirable (cf. (2,3) as well as (5)) to replace $|R_n h_n| \neq O(\|V_n\|_{X^*})$ by $|R_n h_n| \neq O(|V_n h_n|)$, but here we have to confine ourselves to situations where all the values $|V_n h_j|$ are taken into account. Combining this aspect with the UBP, one finally (see [18] for details) obtains the following quantitative extension to the small o-result (14), where ω denotes an abstract modulus of continuity, thus a function, continuous on $[0,\infty)$ such that (cf.[23, p. 96 ff])

(6) $$0 = \omega(0) < \omega(s) \leqslant \omega(s+t) \leqslant \omega(s) + \omega(t) \qquad (0 < s,t) ,$$

which is supposed to satisfy additionally

(7) $$\lim_{t \to 0+} \omega(t)/t = \infty .$$

Theorem: Suppose that for $S_t, T_n, R_n, V_n \in X^*$ *there are a sequence* $\{\varphi_n\} \subset (0,\infty)$, *tending to zero, a positive function* $\sigma(t)$ *on* $(0,\infty)$, *and elements* $h_n \in X$ *satisfying (1) as well as*

(8) $$|S_t h_n| \leqslant C_2 \min\{1, \sigma(t)/\varphi_n\} \qquad (n \in \mathbb{N},\ t > 0) ,$$

(9) $$\limsup_{n\to\infty} |T_n h_n| \geqslant C_3 > 0 ,$$

(10) $$\limsup_{n\to\infty} |R_n h_n| \geqslant C_4 > 0 ,$$

(11) $$|V_n h_n| \leq C_5, \qquad \limsup_{m\to\infty} |V_m h_n|/\varphi_m \leq C_{6,n}\,.$$

Then for each ω *with (6,7) there exists a counterexample* $f_\omega \in X$ satisfying

(12) $$|S_t f_\omega| = O(\omega(\sigma(t))) \qquad (t \to 0+),$$

(13) $$|T_n f_\omega| \neq o(\omega(\varphi_n)) \qquad (n \to \infty),$$

(14) $$|R_n f_\omega| \neq o(|V_n f_\omega|) \qquad (n \to \infty).$$

Thus the negative small o-result (14) (cf. (5)) on the comparison of the processes $\{R_n\},\{V_n\}$ is given in quantitative terms in as much as (12) assures a certain smoothness of the counterexample f_ω, whereas (13) may be interpreted as a precision of its nonsmoothness. In other words, (12) and/or (13) define a class of elements for which by (14) a small o-comparison is violated (cf. (4)).

For a proof of the Theorem (via a gliding hump method) see [8; 18] and the literature cited there. In fact, our approach was very much inspired by Boman's treatment [1] of the problem of S.B. Steckin, mentioned in the beginning. Steckin raised his question on the occasion of the conference on CONSTRUCTIVE FUNCTION THEORY at Blagoevgrad (1977). In contrast to the more theoretical merits of Favard's suggestions in 1963, Steckin posed his problem in the following concrete terms:

Let $C_{2\pi}$ be the Banach space of 2π-periodic, continuous functions on the real axis $\mathbb{R}$, endowed with the usual sup-norm $\|\cdot\|_C$. For $f \in C_{2\pi}$ the error of best approximation by trigonometrical polynomials p_n of degree at most n (i.e., $p_n \in \Pi_n$) is denoted by

(15) $$E_n(f;C_{2\pi}) := \inf\{\|f-p_n\|_C : p_n \in \Pi_n\}$$

and the rth modulus of continuity of the function f by $(r \in \mathbb{N})$

(16) $$\omega_r(t,f;C_{2\pi}) := \sup_{|u|\leq t} \|\Delta_u^r f\|_C\,, \qquad \Delta_u^r f(x) := \sum_{k=0}^{r} \binom{r}{k}(-1)^k f(x+ku)\,.$$

Having established the estimate (cf. [23, p. 331 ff])

$$\omega_r(1/n,f;C_{2\pi}) \leq C_r\, n^{-r} \sum_{k=0}^{n} (k+1)^{r-1}\, E_k(f;C_{2\pi})$$

in 1951, Steckin raised the following question in 1977 (cf. [1]): Does there even exist an estimate of (the more direct) type

(17) $$\omega_r(1/n,f;C_{2\pi}) \leq C_f\, E_n(f;C_{2\pi})$$

for each nondifferentiable function f, say? Clearly, this fits into the frame, envisaged by Favard (cf. (4)). The answer to the particular problem at hand is negative and was given by Boman [1]. Let us show how to regain (a quantitative extension of) Boman 's result from our general Theorem.

Corollary 1: For each ω, *satisfying (6,7), and each positive sequence* $\{M_n\}$, *tending to infinity, there exists a counterexample* $f_0 \in C_{2\pi}$ *(depending on* ω,M*) which satisfies*

$$(18) \qquad \omega_r(t,f_0;C_{2\pi}) \begin{cases} = O(\omega(t^r)) \\ \neq o(\omega(t^r)) \end{cases} \qquad (t \to 0+),$$

but for which nevertheless

$$\omega_r(1/n,f_0;C_{2\pi}) \neq o(M_n E_n(f_0;C_{2\pi})) \qquad (n \to \infty) .$$

Proof: For $f \in C_{2\pi}$ consider the functionals

$$S_t f = \omega_r(t,f;C_{2\pi}), \quad T_n = R_n = S_{1/n}, \quad V_n f = M_n \, E_n(f;C_{2\pi}) .$$

With $h_n(x) = e^{inx}$ one obtains (1,8 - 10) with $\sigma(t) = t^r$, $\varphi_n = n^{-r}$ (and $C_1 = 1$, $C_2 = 2^r$, $C_3 = C_4 = (2/\pi)^r$), using the elementary estimates

$$(19) \qquad |\Delta_u^r h_n(x)| = |1 - e^{inu}|^r \begin{cases} \leq \min\{2^r,(n|u|)^r\} & \text{for } u \in \mathbb{R} \\ \geq (2n|u|/\pi)^r & \text{for } |u| \leq \pi/n . \end{cases}$$

Since $V_m h_n = 0$ for $m \geq n$, condition (11) also holds true with $C_5 = C_{6,n} = 0$. Hence (12 - 14) are valid, completing the proof. □

Note that, choosing $\omega(t) = t^{\beta/r}$, $0 < \beta < r$ (cf. (6,7)), and $M_n = n^\gamma, \gamma > 0$, the parameter β may be arbitrarily small, whereas γ may be arbitrarily large!

The present treatment of Steckin's problem sufficiently exhibits the close interconnection with the more theoretical aspects of Favard's problem. See [8 - 10; 17; 18] for further motivations and details. Here we would like to continue with some more applications, illustrating the wide applicability of the Theorem.

Let us first compare the asymptotic behaviour of the Fourier coefficients

$$(20) \qquad f^\wedge(n) := \frac{1}{2\pi} \int_0^{2\pi} f(u) e^{-inu} \, du$$

of $f \in C_{2\pi}$ with that of the remainders of the n-fold compound trapezoidal rule (Riemann coefficients)

$$(21) \qquad r_n(f) := \frac{1}{n} \sum_{k=1}^{n} f(2\pi k/n) - \frac{1}{2\pi} \int_0^{2\pi} f(u) \, du .$$

In [5] it was mentioned that for "smooth" elements f the coefficients $f^\frown(n)$ and $r_n(f)$ might have the same rate of convergence to zero, whereas otherwise $f^\frown(n)$ would probably converge to zero "faster" than $r_n(f)$. In this connection it was shown in [5] that, given a positive nullsequence $\{\eta_n\}$, there exists a sequence of functions $\{f_n\} \subset C_{2\pi}$ such that

$$|f_n^\frown(n)| \leq \eta_n |r_n(f_n)| \qquad (n \in \mathbb{N}) . \tag{22}$$

With the aid of the general Theorem this may now be strengthened to the following resonance result.

Corollary 2: Given two positive sequences $\{\varepsilon_n\}, \{\eta_n\}$, monotonically decreasing to zero, there exists an even function $f_o \in C_{2\pi}$ (depending on ε, η) such that

$$|f_o^\frown(n)| = O(\varepsilon_n) , \qquad |f_o^\frown(n)| \neq o(\varepsilon_n) \qquad (n \to \infty) , \tag{23}$$

$$\limsup_{n\to\infty} \eta_n |r_n(f_o)| / |f_o^\frown(n)| > 0 . \tag{24}$$

Proof: In $X = C_{2\pi}^e := \{f \in C_{2\pi} : f \text{ even}\}$ consider the elements $h_n(x) = \cos 2nx$ for the functionals (the continuous parameter t being replaced by the discrete one 1/n)

$$S_{1/n} f = \eta_n V_n f = |f^\frown(n)| , \qquad T_n f = |f^\frown(2n)| , \qquad R_n f = |r_n(f)| .$$

Obviously, $R_n h_n = 1$, $T_n h_n = 1/2$, $S_{1/m} h_n = \delta_{m,2n}/2$ (Kronecker δ), and $V_m h_n = \delta_{m,2n}/2\eta_m$. Hence conditions (1,8 - 11) are satisfied for $\sigma(1/n) = \varepsilon_n^2$, $\varphi_n = \varepsilon_{2n}^2$ with $C_1 = C_4 = 1$, $C_2 = C_3 = 1/2$ (note that $\sigma(1/m)/\varphi_n < 1$ only if $2n < m$), $C_5 = C_{6,n} = 0$ so that (12-14) for $\omega(t) = t^{1/2}$ deliver (23,24). □

Note that (24) also implies that there does not exist an estimate of type (cf. (17))

$$|r_n(f)| \leq C_f |f^\frown(n)| \tag{25}$$

for each f belonging to a class, described via (23). On the other hand, there holds true the inequality (cf. [6, p. 111])

$$|r_n(f)| \leq C \sum_{k=1}^{\infty} |f^\frown(kn)| \qquad (f \in C_{2\pi}^e) . \tag{26}$$

Therefore (24) also implies that (26) cannot be strengthened to an estimate of (the more direct) type (25), even not on subclasses (23). In this connection let us mention that, instead of (23), (non)smoothness of the elements may of course be measured by any other appropriate functionals, e.g., by the functionals (15) of best approximation.

Obviously, one may discuss the matter also from the opposite direction. For example, one equally well has the estimate (see [3; 7])

$$|f^\wedge(n)| \leq C \sum_{k=1}^{\infty} |r_{kn}(f)| \qquad (f \in C_{2\pi}^{e})$$

which again cannot be improved to a more direct one of type (25).

<u>*Corollary 3:*</u> *Given two positive sequences* $\{\varepsilon_n\},\{\eta_n\}$, *monotonically decreasing to zero, there exists an even function* $f_o \in C_{2\pi}$ *satisfying (23) as well as*

$$\limsup_{n\to\infty} \eta_n |f_o^\wedge(n)|/|r_n(f_o)| > 0 .$$

<u>Proof:</u> Now consider $h_n(x) = \cos nx - \cos 2nx$ for

$$S_{1/n}f = R_n f = |f^\wedge(n)|, \qquad T_n f = |f^\wedge(2n)|, \qquad V_n f = |r_n(f)|/\eta_n .$$

As in the previous situation conditions (1,8 - 11) are satisfied for $\sigma(1/n) = \varepsilon_n^2$, $\varphi_n = \varepsilon_{2n}^2$ so that the assertions follow for $\omega(t) = t^{1/2}$. □

Some other aspects of an application of the Theorem may be illustrated in connection with the Fejér means (cf. (20))

$$F_{n-1}(f;x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) f^\wedge(k) e^{ikx} \qquad (f \in C_{2\pi}) .$$

In view of the inequality (cf. [22])

$$\|F_{n-1}f - f\|_C \leq \frac{B}{n} \sum_{k=1}^{n} \omega_1(1/k, f; C_{2\pi}) \qquad (f \in C_{2\pi})$$

one may first of all continue with asking for an estimate of type

$$\|F_{n-1}f - f\|_C \leq B_f\, \omega_1(1/n, f; C_{2\pi}) \tag{27}$$

to be valid for some class of functions f. In this situation the general Theorem delivers the following negative result.

<u>*Corollary 4:*</u> *For each* ω, *satisfying (6), and each positive sequence* $\{M_n\}$, *satisfying*

$$\lim_{n\to\infty} M_n = \infty , \qquad \lim_{n\to\infty} M_n/\log n = 0 , \tag{28}$$

there exists a counterexample $f_o \in C_{2\pi}$ *(depending on* ω, M*) such that (18) holds true for* $r = 1$, *but nevertheless*

$$\|F_{n-1}f_0 - f_0\| \neq o(M_n \omega_1(1/n, f_0; C_{2\pi})) \qquad (n \to \infty).$$

Proof: In $X = C_{2\pi}$ consider the elements $h_n(x) = |\sin x| + e^{inx}$ for the functionals

$$S_t f = \omega_1(t, f; C_{2\pi}), \quad T_n f = S_{1/n} f, \quad R_n f = \frac{r_n}{\log r_n} \|F_{r_n - 1} f - f\|_C, \quad V_n f = \frac{r_n M_{r_n}}{\log r_n} S_{1/r_n} f$$

where $r_n \in \mathbb{N}$ is chosen such that (cf. (28))

$$(29) \qquad n \leq r_n, \quad M_{r_n}/\log r_n \leq 1/n.$$

Then $\|h_n\|_C \leq 2$ and $S_t h_n \leq \min\{2\|h_n\|_C, t + nt\} \leq 4\min\{1, tn\}$ so that (8) holds true with $\sigma(t) = t$, $\varphi_n = 1/n$. Concerning (9,10) one has for n, sufficiently large,

$$T_n h_n \geq T_n(e^{inx}) - T_n(|\sin x|) \geq 2/\pi - 1/n \geq C_3 > 0,$$

$$R_n h_n \geq R_n(|\sin x|) - R_n(e^{inx}) \geq B_1 - n/\log r_n \geq C_4 > 0,$$

where we have used (19, 29) and the well-known estimate (for n, sufficiently large, cf. [16, p. 148])

$$\|F_{n-1}(|\sin u|; x) - |\sin x|\|_C \geq B_1 \frac{\log n}{n}.$$

Note that this handles the case $\omega(t) = t$. Moreover, $V_m h_n \leq n M_{r_m}/\log r_m \leq n/m$ by (29) so that (11) follows with $C_5 = 1$, $C_{6,n} = n$. □

The negative result of Corollary 4 has to be contrasted with the following positive statements concerning a direct comparison of type (27): If $f \in C_{2\pi}$ is the boundary value of a function, analytic in the unit disc, that is, has one-sided Fourier series, then indeed there holds true the estimate (cf. [21], also [2])

$$\|F_{n-1}f - f\|_C \leq B\, \omega_1(1/n, f; C_{2\pi}).$$

Moreover, for the spaces $L^p_{2\pi}$, $1 < p < \infty$, one has even generally

$$\|F_{n-1}f - f\|_p \leq B_p\, \omega_1(1/n, f; L^p_{2\pi}) \qquad (f \in L^p_{2\pi}),$$

since $\{-i \operatorname{sgn} k\}$ is a bounded multiplier on these spaces in view of a theorem of M. Riesz. Corresponding statements may of course also be given for the Abel-Poisson means, either by a separate treatment or by using the fact that the Fejér and Abel-Poisson means are equivalent (global comparison, cf. [4, p. 495]). See [21] (and the literature cited there) for a discussion in connection with the different behaviour of harmonic and analytic functions near the boundary.

Our last application is concerned with a first contribution to the classical problem of nonequiconvergence, by the way illustrating the fact that, if some of the functionals are omitted in the Theorem, the conditions upon the remaining functionals still imply the relevant assertions.

For $f \in C_{2\pi}$ consider the Fourier partial sums (cf. (20))

$$(s_n f)(x) := \sum_{k=-n}^{n} f^{\wedge}(k) e^{ikx} := \frac{1}{2\pi} \int_0^{2\pi} f(u)\, D_n(x-u)\, du ,$$

which are to be compared with the trigonometric Lagrange polynomials

$$(\Lambda_n f)(x) := \frac{1}{2n+1} \sum_{k=0}^{2n} f(x_{kn})\, D_n(x - x_{kn}) ,$$

$$D_n(u) := \frac{\sin(n+1/2)u}{\sin(u/2)} , \qquad x_{kn} := \frac{2k\pi}{2n+1} , \qquad 0 \leq k \leq 2n .$$

Contrasting to the fact that both processes have asymptotically the same behaviour on e.g. Lipschitz classes, Faber [12, pp. 431 - 438] observed in 1910 that there are continuous functions for which one process converges uniformly, but for which the other diverges everywhere (for extensions see [11; 19; 20] and the literature cited there). In this connection an application of the Theorem delivers the following quantitative result on the nonequiconvergence in the uniform topology.

Corollary 5: For each $r \in \mathbb{N}$ *and* ω*, satisfying (6,7), there exists a counterexample* $f_0 \in C_{2\pi}$ *(depending on* r, ω*) such that*

$$\| s_n f_0 - f_0 \|_C = O(\omega(n^{-r} \log n)) ,$$

$$\| \Lambda_n f_0 - f_0 \|_C \neq o(\omega((n \log n)^{-r}) \log n) .$$

Proof: Using ideas of [11], let H be a function, infinitely often differentiable on $\mathbb{R}$, satisfying

$$0 \leq H(u) \leq 1 , \qquad H(0) = 1, \qquad H(u) = 0 \text{ for } |u| \geq 1.$$

Then, first of all, for each $A, x \in \mathbb{R}$

$$(30) \quad \left| \int_{-1}^{1} H(u) \frac{\sin A(x-u)}{x-u} du \right| \leq |H(x)| \left| \int_{-1}^{1} \frac{\sin A(x-u)}{x-u} du \right| + 2\| H' \|_C \leq C .$$

Let us apply the Theorem to $X = C_{2\pi}$,

$$S_{1/n} f = \| s_n f - f \|_C , \qquad T_n f = \| \Lambda_n f - f \|_C / \log n ,$$

$$h_n(x) = \sum_{k=o}^{2n} [\operatorname{sgn} D_n(\pi - x_{kn})]\, H((n \log n)(x - x_{kn})) ,$$

where $n \in \mathbb{N}$ is already chosen so large that $1/n \log n \leqslant \pi/2(2n+1)$. Then h_n has compact support in $-\pi/(2n+1) \leqslant x \leqslant 2\pi - \pi/(2n+1)$ so that h_n may be considered as an element of $C_{2\pi}$ satisfying $\|h_n\|_C = 1$ and $h_n(x_{kn}) = \operatorname{sgn} D_n(\pi - x_{kn})$. Consequently,

$$\|\Lambda_n h_n\|_C \geqslant |(\Lambda_n h_n)(\pi)| = \frac{1}{2n+1} \sum_{k=o}^{2n} |D_n(\pi - x_{kn})| \geqslant C \log n$$

which establishes (9). Moreover,

$$|(s_j h_n)(x)| \leqslant \frac{1}{\pi} \sum_{k=o}^{2n} B_k + C ,$$

$$B_k := \Big| \int_{x_{kn}-1/n \log n}^{x_{kn}+1/n \log n} H((n \log n)(u - x_{kn}))\, \frac{\sin(n+1/2)(x-u)}{x-u}\, du \Big| .$$

Since for $-\pi/(2n+1) \leqslant x \leqslant 2\pi - \pi/(2n+1)$ there exists $0 \leqslant j \leqslant 2n$ such that $|x - x_{jn}| \leqslant \pi/(2n+1)$, one has for $k \neq j$ and $|u - x_{kn}| \leqslant 1/n \log n \leqslant \pi/2(2n+1)$ that

$$|x-u| \geqslant |x_{kn} - x_{jn}| - |x - x_{jn}| - |u - x_{kn}| \geqslant \frac{1}{2}\, \frac{\pi |j-k|}{2n+1} ,$$

$$\sum_{\substack{k=o \\ k \neq j}}^{2n} B_k \leqslant \sum_{\substack{k=o \\ k \neq j}}^{2n} \int_{x_{kn}-1/n \log n}^{x_{kn}+1/n \log n} \frac{du}{|x-u|} \leqslant \frac{4}{\pi n \log n} \sum_{\substack{k=o \\ k \neq j}}^{2n} \frac{2n+1}{|j-k|} \leqslant C .$$

For $k = j$ substitute $v = (n \log n)(u - x_{kn})$ to obtain by means of (30)

$$B_j = \Big| \int_{-1}^{1} H(v)\, \frac{\sin \frac{n+1/2}{n \log n}(v - (x - x_{jn}) n \log n)}{v - (x - x_{jn}) n \log n}\, dv \Big| \leqslant C .$$

Thus $S_{1/j} h_n \leqslant C$, uniformly for $j, n \in \mathbb{N}$, and (8) follows with $\sigma(1/n) = n^{-r} \log n$, $\varphi_n = (n \log n)^{-r}$ because

$$S_{1/j} h_n \leqslant C (j^{-r} \log j) \|h_n^{(r)}\|_C \leqslant C\, \sigma(1/j)/\varphi_n .$$

Since conditions (1,8,9) imply the assertions (12,13), the result follows. □

To mention some specific situations, let $r = 1$ and ω be a modulus of continuity which behaves like $1/|\log t|^{1/2}$ near the origin (cf. [23, p. 97]). Then Corollary 5 assures the existence of a counterexample $f_o \in C_{2\pi}$ for which

$$\|s_n f_o - f_o\|_C = \mathcal{O}((\log n)^{-1/2}), \qquad \|\Lambda_n f_o - f_o\|_C \neq o((\log n)^{1/2}) .$$

On the other hand, for $r \in \mathbb{N}$ and $\omega(t) = t^{\alpha}$, $0 < \alpha < 1/(r+1)$, there exists $g_0 \in C_{2\pi}$ such that (cf. (5))

$$\frac{\|\Lambda_n g_0 - g_0\|_C}{\|s_n g_0 - g_0\|_C} \neq o((\log n)^{1-\alpha(r+1)}) \neq O(1) \qquad (n \to \infty) .$$

Obviously, the same kind of analysis yields results on the nonequiconvergence at a prescribed point, or even at a denumerable set of points upon using an additional condensation argument (cf. [18]). Phenomena on the nonequiconvergence at each point of an interval, however, are beyond the scope of the Theorem.

References

[1] J. Boman. A problem of Steckin on trigonometric approximation. In: Constructive Function Theory '77 (Proc. Conf. Blagoevgrad, 1977, Eds. B. Sendov and D.Vacov). Publ. House Bulg. Acad. Sci., Sofia, 1980, 269 - 273.

[2] J. Boman. Equivalence of generalized moduli of continuity. Ark. Mat. 18(1980), 73 - 100.

[3] H. Brass. Umkehrsätze beim Trapezverfahren. Aequationes Math. 18(1978),338-344.

[4] P.L. Butzer and R.J. Nessel. Fourier Analysis and Approximation, Vol. I. Birkhäuser, Basel, and Academic Press, New York, 1971.

[5] C.-H. Ching and C.K. Chui. Asymptotic similarities of Fourier and Riemann coefficients. J. Approx. Theory 10(1974), 295 - 300.

[6] P.J. Davis and P. Rabinowitz. Numerical Integration. Blaisdell, Waltham, Mass., 1967.

[7] W. Dickmeis. Comparison theorems for compound quadrature formulas (in print).

[8] W. Dickmeis, R.J. Nessel, and E. van Wickeren. On the sharpness of estimates in terms of averages. Math. Nachr. 117(1984) (in print).

[9] W. Dickmeis, R.J. Nessel, and E. van Wickeren. A general approach to quantitative negative results in approximation theory. (in print).

[10] W. Dickmeis, R.J. Nessel, and E. van Wickeren. A general approach to counterexamples in numerical analysis. Numer. Math. 43(1984), 249 - 263.

[11] P. Erdös and G. Grünwald. Über einen Faber'schen Satz. Ann. of Math. 39(1938), 257 - 261.

[12] G. Faber. Über stetige Funktionen (zweite Abhandlung). Math. Ann. 69(1910), 372 - 443.

[13] J. Favard. Sur la comparison des procédés de sommation. In: On Approximation Theory (Proc. Conf. Oberwolfach, 1963, Eds. P.L. Butzer and J. Korevaar, ISNM 5). Birkhäuser, Basel, 1964, 4 - 11.

[14] J. Favard. On the comparison of the processes of summation. SIAM J. Numer. Anal. 1(1964), 38-52.

[15] S. Kaczmarz and H. Steinhaus. Theorie der Orthogonalreihen. Chelsea Publ., New York, 1951.

[16] I.P. Natanson. Konstruktive Funktionentheorie. Akad. Verl., Berlin, 1955.

[17] R.J. Nessel and E. van Wickeren. Some negative results in connection with Marchaud-type inequalities. In: General Inequalities 4 (Proc. Conf. Oberwolfach, 1983). Birkhäuser, Basel. (in print)

[18] R.J. Nessel and E. van Wickeren. Negative results in connection with Favard's problem on the comparison of approximation processes. In: Anniversary Volume on Approximation Theory and Functional Analysis (Proc. Conf. Oberwolfach, 1983, Eds. P.L. Butzer et al., ISNM 65). Birkhäuser, Basel, 1984 (in print).

[19] A.A. Privalov. Approximation of functions by interpolation polynomials. In: Fourier Analysis and Approximation Theory (Proc. Conf. Budapest, 1976, Eds. G. Alexists and P. Turán). North-Holland, Amsterdam, 1978, 659 - 670.

[20] A.A. Privalov. Lagrange interpolation polynomials and orthogonal Fourier-Jacobi series (Russian). Mat. Zametki 20(1976), 215 - 226 = Math. Notes 20(1976), 679-685.

[21] H.S. Shapiro. The modulus of continuity of an analytic function. In: Spaces of Analytic Functions. Lecture Notes in Math. 512(1976), 131 - 138.

[22] S.B. Steckin. The approximation of periodic functions by Fejér sums (Russian). Trudy Mat. Inst. Steklov 62(1961), 48-60 = Amer. Math. Soc. Transl. (2) 28 (1963), 269 - 282.

[23] A.F. Timan: Theory of Approximation of Functions of a Real Variable. Pergamon Press, New York, 1963.

Lehrstuhl A für Mathematik
RWTH Aachen
Templergraben 55

D - 5100 Aachen