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DERIVATIVES OF TRIGONOMETRIC POLYNOMIAL APPROXIMATION

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For a function in a Banach space of 2π periodic functions, the asymptotic behaviour of the derivatives of some n-th degree trigonometric polynomial approximations to f and the smoothness of f will be related. For T_n, the best n-th degree trigonometric polynomial approximation in $B = L_p[0,2\pi]$ or $B = C[0,2\pi]$, it was shown by Zamanski [6] and Sunouchi [5] that $\|T_n^{(r)}\|_B \le Kn^{r-\alpha}$ for $r > \alpha$ is equivalent to $\|\Delta_h^r f\|_B \le Lh^\alpha$ where $\Delta_h f(x) = f(x+h) - f(x)$ and $\Delta_h^k f = \Delta_h(\Delta_h^{k-1} f)$. It was further noticed [3, p. 569] that the above result remains valid if B is any Banach space of 2π periodic functions for which $\|f(\cdot+a)\|_B = \|f(\cdot)\|_B$ and $\|\Delta_h f\|_B = o(1)$ as $h \to 0$. Here we will observe that $\alpha < r$ can be replaced by $\alpha \le r$ and that we have $\|T_n^{(r)}\| = O(1)$ if and only if $\|\Delta_h^r f\| = O(h^r)$ while we recall $E_n(f) = O(n^{-r})$ if and only if $\|\Delta_h^{r+1} f\| = O(h^r)$. In fact probably the reason why $\alpha = r$ was not treated in [5] is that comparison was made to $E_n(f)$ rather than to $\|\Delta_h^r f\|$. We will investigate similar equivalence relations for trigometric approximation processes which are linear and generated by convolutions. In particular we find that some approximation processes that will not register improvement in the rate of convergence for smoothness much lower than $\mathrm{Lip}^*\alpha$ will still have a condition on the r-th derivative that implies $\mathrm{Lip}^*\alpha$. One should note that in many results on the subject (see [2] and the textbook [1]) the asymptotic behaviour of the derivative is tied to the behaviour of the rate of convergence of the approximation process. The result here for $\alpha < r$ follows a result of mine [4] about convolution kernels which are not necessarily trigonometric polynomials. As the kernel here is a trigonometric polynomial, a simpler proof is possible. We will also treat the case $\alpha = r$ and prove the result for a wider variety of spaces. It can be noted that the result for derivatives of these approximation processes is valid for spaces for which the Sunouchi-Zamanski result for polynomials of best approximation is not.

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2. Remarks on Sunouchi's and Zamanski's theorems. The following is a summation and a small modification of the Sunouchi-Zamanski result and its generalization to some Banach spaces ([3], [5] and [6]).

Theorem 2.1. Suppose B is a Banach space of functions on $T = [0,2\pi]$ satisfying $\|(f\cdot+h)-f(\cdot)\|_B = o(1)$ and $\|f(\cdot+a)\| = \|f(\cdot)\|$ and $\phi(n)$ a decreasing sequence satisfying $n^r\phi(n) \geq A > 0$ for some r and $\lim_{n\to\infty} n^\delta\phi(n) = 0$ for some δ, then if for trigonometric polynomial Q_n, $Q_n \in B$ implies $Q_n' \in B$ we have

(a) $\|T_n^{(r)}\|_B \leq Mn^r\phi(n)$ is equivalent to $\|\Delta_h^r f\|_B \leq k\phi([\frac{1}{|h|}])$,

and

(b) $E_n(f,B) \leq M\phi(n)$ is equivalent to $\|\Delta_h^{r+1} f\|_B \leq k\phi([\frac{1}{|h|}])$,

where T_n is the best trigonometric polynomial approximation to f in B and $E_n(f,B) \equiv \|f-T_n\|_B$.

Remark 2.2. Note that for $\phi(n) = n^{-r}$ $\|T_n^{(r)}\|_B \leq M$ is equivalent to $\|\Delta_h^r f\|_B \leq M_1 h^r$ (the present observation), but $\|E_n(f,B)\| \leq Mn^{-r}$ is equivalent to $\|\Delta_h^{r+1} f\|_B \leq M_1 h^r$ and in most cases this connection to higher difference is covered by resorting to infinite sums. ((b) is state here mainly for compaison with (a)).

Proof. Using [3] for instance we have that the inequality $\|\Delta_h^r f\|_B \leq K\phi([\frac{1}{|h|}])$ or $\|\Delta_h^{r+1} f\|_B \leq L\phi([\frac{1}{|h|}])$ implies $\|E_n(f,B)\| \leq M\phi(n)$ (for any decreasing $\phi(n)$). We will show that $E_n(f,B) \leq M\phi(n)$ implies $\|T_n^{(r+2)}\| \leq M_1 n^{r+2}\phi(n)$. For this we use the standard technique of estimating $T_{2^k} - C = \sum_{\ell=1}^{k} T_{2^\ell} - T_{2^{\ell-1}}$, and using $n^r\phi(n) \geq A$, we have $\|T_{2^k}^{(r+2)}\| \sim 2^{k(r+2)}\phi(2^k)$ and therefore $\|T_n^{(r+2)}\| \sim n^{r+2}\phi(n)$. To show that $\|T_n^{(r)}\| \sim n^r\phi(n)$ we show this first for $B = C[0,2\pi]$. For $B = C$ we have $|T_n^{(r+1)}(\xi)| = h^{r+1}|\Delta_h^{r+1} T_n(x)| \leq h^{r+1}|\Delta_h^{r+1}(T_n(x)-f(x))| + h^{r+1}|\Delta_h^{r+1} f(x)|$ $\leq Mh^{r+1}2^{r+1}\phi(n) + Mh^{r+1}\phi([\frac{1}{|h|}])$, and setting $h = \frac{1}{n}$, we have for any x a point ξ between x and $x + rh$ such that $|T_n^{(r+1)}(\xi)| \leq M_2 n^{r+1}\phi(n)$. Using $\|T_n^{(r+2)}\| \leq Mn^{r+2}\phi(n)$, we have for $h \sim \frac{1}{n}$ $|T_n^{(r+1)}(x)| \leq |T_n^{(r+1)}(x) - T_n^{(r+1)}(\xi)|$ $+ |T_n^{(r+1)}(\xi)| \leq rh|T_n^{(r+2)}(\eta)| + |T_n^{(r+1)}(\xi)| \leq M_3 n^{r+1}\phi(n)$ and similarly we estimate $T_n^{(r)}(x)$. Having our result for $C[0,2\pi]$, we follow [3] and observe that if $\|f - T_n\|_B \leq M\phi(n)$ and $\|\Delta_h^r f\|_B \leq M\phi([\frac{1}{|h|}])$, then for $g \in B^*$ $\|g\|_{B^*} = 1$, we define $F = f*g \equiv \langle f(x+\cdot),g(\cdot)\rangle$ and $Q_n = T_n*g = \langle T_n(x+\cdot),g(\cdot)\rangle$, and therefore $\|Q_n^{(r+2)}\|_{C[0,2\pi]} \leq Mn^{r+2}\phi(n)$, $\|Q_n - F\| \leq M\phi(n)$ and $\|\Delta_h^r F\| \leq M\phi([\frac{1}{|h|}])$.

This now implies $\|Q_n^{(r)}\|_{C[0,2\pi]} \le Mn^r\phi(n)$ and therefore $|Q_n^{(r)}(0)| \le Mn^r\phi(n)$, and since we have the choice of g (as long as g is in B^* and $\|g\|_{B^*} = 1$), $\|T_n^{(r)}\|_B \le Mn^r\phi(n)$. To prove the other direction of (a), we use the techniques of Sunouchi's result following verbatim the proof for L_p [5, p. 234-5], as what is used there is that the sequence $E_n(f,B)$ is decreasing and tends to zero which is valid for all the spaces in this theorem. We obtain $E_n(f,B) \le A \sum_{k=0}^{\infty} (2^k n)^{-r} \|T^{(r)}_{2^{k+1}n}\|_B$ and using $n^\delta\phi(n)$ tends to zero, $E_n(f,B) \le K\phi(n)$. We can now write

$$\|\Delta_h^r f\| \le \|\Delta_h^r(f-T_n)\|_B + \|\Delta_h^r T_n\|_B \le 2^r E_n(f,B) + h^r\|T_n^{(r)}\|_B \le M_1\phi([\tfrac{1}{|h|}]).$$

We use the earlier observation that $E_n(f,B) \le M\phi(n)$ implies under the condition $n^r\phi(n) \ge A > 0$, that $\|T_n^{(r+1)}\|_B \le Mh^{r+1}\phi(n)$ and this implies $\|\Delta_h^{r+1} f\|_B \le M\phi([\frac{1}{|h|}])$ which completes the proof of (b).

<u>Remark 2.3</u>. It should be noted that the condition $\|f(\cdot+h)-f(\cdot)\|_B = o(1)$ for all $f \in B$ cannot be disposed of. Take for instance the space $L_\infty[0,2\pi]$ and the function $f(x)$ given by $f(x) = 1 + \sin x$ on $0 < x < \pi$ and $f(x) = -1 + \sin x$ on $\pi < x < 2\pi$. It is clear that $T_n = \sin x$ and therefore $\|T_n'\| = c$ but $\|\Delta_h f\|_{L_\infty} \ne O(h)$.

<u>Remark 2.4</u>. It will later be shown that for linear trigonometric approximation processes, an analogue of the Sunouchi-Zamanski result is valid for spaces of measures and other subspaces of S'. Here a counter-example is evident with the function $\alpha(x) = 1 + \sin x$ for $0 \le x < \pi$ and $\alpha(x) = -1 + \sin x$ for $\pi < x \le 2\pi$ where $\alpha \in$ B.V. and $T_n = \sin x$ (not unique). We see that for polynomials of best approximation the Sunouchi-Zamanski result cannot be extended to $B =$ B.V.$[0,2\pi]$.

<u>Remark 2.5</u>. If $n^{r-\delta}\phi(n) \ge A > 0$ for some $\delta > 0$ will replace $n^r\phi(n) \ge A > 0$ in Theorem 2.1, we have $E_n(f,B) \le M\phi(n)$ is equivalent to $\|\Delta_h^r f\| \le K\phi([\frac{1}{|h|}])$.

3. <u>The result for linear convolution approximation processes on $C[0,2\pi]$</u>. Our initial result will be stated and proved on $C[0,2\pi]$.

<u>Theorem 3.1</u>. Let $K_n(t)$ be a sequence of n-th degree trigonometric polynomials satisfying (a) $\int_{-\pi}^{\pi} K_n(t)dt = 1$, (b) $\int_{-\pi}^{\pi} |K_n(t)|dt \le M$ and (c) $\int_{-\pi}^{\pi} |t|^\beta |K_n(t)|dt \le M_1 n^{-\beta}$ for some $\beta>0$; then $\| \frac{d^r}{dx^r} (K_n * f(x))\|_{C[0,2\pi]} \le Kn^{r-\alpha}$ for $r \ge \alpha$ (where $K_n * f(x) \equiv \int_{-\pi}^{\pi} K_n(x-t)f(t)dt$) is equivalent to

$\|\Delta_h^r f(x)\|_{C[0,2\pi]} \le Lh^\alpha$.

We can use for $r > \alpha$ the result B of theorem 5.1 of [4] as $k = 1$ will be sufficient since $\|K_n\|_{L_1} \le M$ implies via Bernstein's theorem $\|K_n^{(r)}\|_{L_1} \le Mn^r$. For $r = \alpha$ we have some additional difficulty which could have been overcome in [4] too but was not. We will show that in the case dealt with here some of the effort in Theorem 5.1 of [4] is not needed.

Proof Assuming $\|\Delta_h^r f\|_C \le Lh^\alpha$, we have $\|f-T_n\|_C \le M_2 n^{-\alpha}$ for T_n, the best n-th degree trigonometric polynomial approximation to f in C. Therefore, using (b), $\|K_n*(f-T_n)\|_{C[0,2\pi]} \le MM_2 n^{-\alpha}$, and using Bernstein's inequality, $\|(\frac{d}{dx})^r[K_n*(f-T_n)]\| \le MM_2 n^{r-\alpha}$ (as a convolution with an n-th degree trigonometric polynomial is also such a polynomial). Using Zamanski's result [6], for $\alpha < r$, $\|T_n^{(r)}\| \le M_3 n^{r-\alpha}$ and therefore $\|(\frac{d}{dx})^r(K_n*T_n)\| = \|K_n*T_n^{(r)}\| \le MM_3 n^{r-\alpha}$ which implies $\|(\frac{d}{dx})^r K_n*f\| \le M(M_2+M_3)n^{r-\alpha}$. For $\alpha = r$ we recall than an n-th degree trigonometric polynomial, A_n, exists such that $\|A_n\|_1 \le M$ and $\|A_n*f-f\| \le M\omega_r(f,1/n)$; therefore, $\|(\frac{d}{dx})^r A_n*f\| = \|\lim_{h\to 0} \frac{1}{h^r}\Delta_h^r(A_n*f)(x)\|$ $\le \sup_h \|\frac{1}{h^r}\Delta_h^r(A_n*f)\| \le \sup_h \frac{1}{h^r}\|A_n*\Delta_h^r f\| \le N\cdot L$. We now use $K_n*(I-A_n)*f$ instead of $K_n*(f-T_n)$. One can note that for this part of the proof conditions (a) and (c) are redundant.

To prove that $\|(\frac{d}{dx})^r K_n*f\|_C \le Kn^{r-\alpha}$ implies $\|\Delta_h^r f\| \le Lh^\alpha$, we will need the following Lemma:

Lemma 3.2. Suppose $K_n(t)$ is a periodic function satisfying (a), (b) and (c) of Theorem 1, and $L_n(t) = \sum_{\ell=1}^{m}(-1)^{\ell+1}\binom{m}{\ell}K_{n,\ell}(t)$ where $K_{n,1}(t) \equiv K_n(t)$ and $K_{n,\ell}(t) \equiv K_n*K_{n,\ell-1}(t)$, and m is such that $m\cdot\min(\beta,1) \ge r$; then $\|\Delta_h^r f\| \le Lh^\alpha$ for $\alpha \le r$ implies $\|L_n*f-f\| \le B\,M_1^m Ln^{-\alpha}$ where B depends only on m and L.

Proof This result is actually part B of Theorem 3.1 in my paper [2, p. 54-55], where one writes n^{-1} for σ_n, K_n for G_n and L_n for $I-(I-A_n)^m$. Actually the proof of that part is neither long nor complicated but I feel it would be too repetative to include here.

To complete the proof of Theorem 3.1, we observe that $L_n(t)$ is a trigonometric polynomial of the same degrees as $K_n(t)$ and that $\|(\frac{d}{dx})^r K_{n,\ell}*f\| = \|K_{n,\ell-1}*(K_n*f)^{(r)}\| \le M^{\ell-1}Kn^{r-\alpha}$ and therefore $\|(L_n*f)^{(r)}\| \le (1+M)^m\, Kn^{r-\alpha}$. L_n*f converges to f since K_n*f does. One can write $\|L_n*f-L_{2n}*f\| \le \|L_n*f-L_{2n}*L_n*f\| + \|L_n*L_{2n}*f-L_{2n}*f\| \le BM_1{}^m[(2n)^{-r}\|(L_n*f)^{(r)}\| +$ $n^{-r}\|(L_{2n}*f)^{(r)}\|] \le M_3 n^{-\alpha}$. Consequently, $\|L_n*f-f\| \le \sum_{\ell=0}^{\infty} \|L_{2^\ell n}*f-L_{2^{\ell+1}n}*f\|$ $\le \frac{M_3}{n^\alpha}\sum_{\ell=0}^{\infty} 2^{-\ell\alpha} \le M_4 n^{-\alpha}$. But, $\|\Delta_h^r f\| \le \|\Delta_h^r(f-L_n*f)\| + \|\Delta_h^r(L_n*f)\| \le 2^r M_4 n^{-\alpha}$ $+ h^r M_5 n^{r-\alpha}$ which implies $\|\Delta_h^r f\| \le Lh^\alpha$ when we choose $n = [\frac{1}{h}]$.

4. <u>The analogous result for other spaces</u>. We can now show that Theorem 3.1 is not exclusive to $C[0,2\pi]$.

<u>Theorem 4.1</u> For a Banach space B of 2π periodic functions satisfying $\|f(\cdot)\|_B = \|f(\cdot+a)\|_B$ and $\|\Delta_h f(\cdot)\|_B = o(1)$ as $h \to 0$, and $K_n(t)$ of Theorem 3.1, the inequalities $\|(K_n*f)^{(r)}\|_B \le M\, n^{r-\alpha}$ and $\|\Delta_h^r f\|_B \le Lh^\alpha$ are equivalent for $r \ge \alpha$.

<u>Theorem 4.2</u> Suppose B is a Banach space which is continuously imbedded in S' (the space of tempered distributions, and for which $\|f(\cdot)\|_B = \|f(\cdot+a)\|_B$, all $T_n \in B$, and $B = X^*$ where X is a Banach space of functions such that $\|\Delta_h g(\cdot)\|_X = o(1)$, $h \to 0$ for all $g \in X$. Then for $K_n(t)$ of Theorem 3.1 the result of Theorem 4.1 is valid where $(K_n*f)^{(r)}$ is the formal derivative as a trigonometric polynomial.

<u>Proof</u> We choose $g \in B^*(B^*$ the dual space to B) with $\|g\|_{B^*} = 1$ or $g \in X$ (B the dual spaces to X) with $\|g\|_X = 1$. For $\|\Delta_h^r f\|_B \le Lh^\alpha$ we define $F(x) = \langle f(x-\cdot), g(\cdot)\rangle$ and observe that $F \in C[0,2\pi]$ and moreover $\|\Delta_h^r F\|_{C[0,2\pi]} \le Lh^\alpha$. Therefore, $\|(K_n*F)^{(r)}\|_C \le Mn^{r-\alpha}$ and hence $|(K_n*F)^{(r)}(0)| \le Mn^{r-\alpha}$. But $(K_n*F)^{(r)}(x) = (K_n*f)^{(r)}*g$ as the r derivative of K_n*f exists in the strong or weak* sense for Theorems 4.1 and 4.2 respectively and is equal to $(K_n*f)^{(r)}(x)$. Therefore we can choose g so that $|(K_n*F)^{(r)}(0)|$ is as close as we like to $\|(K_n*f)^{(r)}\|_B$, which implies $\|(K_n*f)^{(r)}\|_B \le Mn^{r-\alpha}$. Assume $\|(K_n*f)^{(r)}\|_B \le Mn^{r-\alpha}$, and consequently $\|(K_n*F)^{(r)}\|_C \le \|(K_n*f*g)^{(r)}\|_C$ $\le \|(K_n*f)^{(r)}\|_B \le Mn^{r-\alpha}$, which in turn will imply $|\Delta_h^r F(0)| \le \|\Delta_h^r F\| \le Lh^\alpha$. The last inequality with an appropriate choice of g will then yield $\|\Delta_h^r f\| \le Lh^\alpha$.

<u>Remark 4.3</u> Theorem 4.2 applies to the spaces $B = L_\infty[0,2\pi]$ and $B.V[0,2\pi]$ to which, according to examples in remarks 2.3 and 2.4, the Sunouchi-Zamanski result cannot be generalized. Theorem 4.2 applies to duals of Sobolev and Besov spaces as well. In Theorem 2.1 we could assume $T_n \in B$ for all T_n but as it is

stated it applies to the spaces H_p as well.

5. Remarks and applications. We can now mention some of the approximation operators to which Theorems 3.1, 4.1 and 4.2 are applicable:

I. The Fejér sum (Cesàro summability of trigonometric partial sums) given by $\sigma_n(f,x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt$. (In this case $\beta < 1$ and $M = 1$).

II. The La Vallée Poussin sum $\tau_n(f,x) = 2\sigma_{2n}(f,x) - \sigma_n(f,x)$. (In this case the degree of the trigonometric polynomial is $2n$, $\beta < 1$ and $M = 3$).

III. The Jackson operator $J_n(f,x) = \lambda_n^{-1}\int_{-\pi}^{\pi} f(x-t)\left(\frac{\sin nt/2}{\sin t/2}\right)^{2r} dt$ where λ_n is given by $J_n(1,x) = 1$. (In this case the degree is rn, $\beta < 2r - 1$ and $M = 1$).

We note that the results applied to the above approximation processes would be valid for all r and for spaces B as in theorems 4.1 and 4.2 (see also Remark 4.3).

We recall that the partial sum of the trigonometric expansion, $S_n f$ $\left(S_n f = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sin(2n+1)t/2}{\sin t/2}\,dt\right)$ satisfies for $1 < p < \infty$ $\|S_n f\|_p \le K_p \|f\|_p$ and $\|S_n f - f\|_p \le (K_p+1)\|f - T_n\|_p = (K_p+1)E_n(f,L_p)$ where K_p tends to infinity as p tends to 1 or ∞) . Hence, perhaps as a final remark, we have:

Theorem 5.1 For $1 < p < \infty$, $\|(S_n f)^{(r)}\|_p \le Mn^{r-\alpha}$ for $r \ge \alpha$ if and only if $\|\Delta_h^r f\|_p \le Lh^\alpha$.

Proof For $\|\Delta_h^r f\|_p \le Lh^\alpha$, $\|S_n f - T_n\|_p \le \|S_n(f-T_n)\| \le K_p\|f-T_n\|_p \le K_p M_1 n^{-\alpha}$ and therefore $\|(S_n f - T_n)^{(r)}\|_p \le K_p M_1 n^{r-\alpha}$, and as $\|T_n^{(r)}\|_p \le M_2 n^{r-\alpha}$ we have $\|(S_n f)^{(r)}\|_p \le M_n^{r-\alpha}$. Similarly, if $\|(S_n f)^{(r)}\|_p \le Mn^{r-\alpha}$, we have

$$\|S_n f - S_{2n} f\|_p = \|S_n(S_{2n} f) - S_{2n} f\| \le (K_p+1)E_n(S_{2n} f, L_p) \le Kn^{-r}M(2n)^{r-\alpha} \le An^{-\alpha},$$

and since in L_p $S_n f \to f$, we have $\|T_n f - f\|_p \le \|S_n f - f\|_p \le \sum_{\ell=0}^{\infty} \|S_{2^\ell n} f - S_{2^{\ell+1} n} f\|_p \le Kn^{-\alpha}$. Using $\|S_n f - f\| \le Kn^{-\alpha}$, our theorem (or should we say remark) follows.

We use only $\|S_n f\|_B \le K\|f\|_B$ and our theorem is valid for all those spaces, called also spaces admitting convergence.

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