КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 291—297

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 291-297

ON THE HAUSDORFF CONVERGENCE OF SUMMATIVE FORMULAE

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1. Introduction. Let $x_{i,n}$, $0 \le i \le n$, $n=1,2,\dots$, be points in the interval $\Delta=[a,b]$ and let φ_i^n, $0 \le i \le n$, $n=1,2,\dots$, be positive continuous functions defined in Δ such that for all $x \in [a,b]$

$$\sum_{k=0}^{n} \varphi_k^n(x)=1.$$

A sequence $\{\Phi_n\}_{n=1}^{\infty}$ of summative formulae is a sequence of linear positive operators on B_Δ defined by

$$\Phi_n(f;x)=\sum_{i=0}^{n} f(x_{i,n})\varphi_i^n(x),$$

where $f \in B_\Delta$ and B_Δ is the set of all functions defined in Δ.

Well known examples of summative formulae are Bernstein polynomials ($\Delta=[0,1]$, $x_{i,n}=i/n$, $\varphi_i^n=\binom{n}{i}x^i(1-x)^{n-i}$) and Fejer interpolative polynomials ($\Delta=[-1,1]$, $x_{i,n}=\cos(\pi(2i-1)/(2n))$, and $\varphi_i^n=(\cos(n \arccos x)/n(x-x_{i,n}))^2(1-xx_{i,n})$).

Given a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of summative formulae, it is important to determine large classes of functions f, for which the sequence of functions $\{\Phi_n(f)\}_{n=1}^{\infty}$ is convergent to f (with respect to a specified metric). It has been proved that both Bernstein and Fejer polynomials are convergent with respect to the Hausdorff distance, if f is a Hausdorff continuous (H-continuous) function; see [1].

In this paper we investigate the inverse problem: wheather from the convergence of $\{\Phi_n(f)\}$ to f for some function f, the H-continuity of f follows. For the case of the Bernstein polynomials a negative answer can be obtained easily by considering the function

$$f(x) = \begin{cases} 0 \text{ for } x\in[0,1) \\ 1 \text{ for } x=1. \end{cases}$$

In this case, however, f is not H-continuous in a single point (x=1) and this point is an endpoint of the interval.

Here we prove a stronger result, namely the correctness of the following claim:

Theorem. If $\{\Phi_n\}$ is an arbitrary sequence of summative formulae with $\Delta=[0,1]$ that is H-convergent for H-continuous functions, then there exists a function f such that $\{\Phi_n(f)\}$ is H-convergent to f and the completed graph of f is the set $I^2=[0,1]\times[0,1]$ (the unit square).

2. Preliminary statements. For each subinterval $[c,d]$ of Δ define

$$\bar{\delta}_{c,d}(x) = \begin{cases} 1 \text{ if } x\in[c,d], \\ 0 \text{ if } x\in[0,1]\setminus[c,d]; \end{cases}$$

$$\underline{\delta}_{c,d}(x) = 1-\bar{\delta}_{c,d}(x).$$

Lemma 1. If $0\leq c<c_1<d_1<d\leq 1$ then

$$\inf_{x\in[c_1,d_1]} \Phi_n(\bar{\delta}_{c,d};x) \xrightarrow[n\to\infty]{} 1, \quad \sup_{x\in[c_1,d_1]} \Phi_n(\underline{\delta}_{c,d};x) \xrightarrow[n\to\infty]{} 0.$$

Proof. Follows from the H-continuouty of $\bar{\delta}_{c,d}$ and $\underline{\delta}_{c,d}$. □

Denote $L_x = \{(i,n): 0\leq i\leq n,\ x_{i,n}=x\}$.

Lemma 2. Let $0<a_1<x_0<b_1<1$, $\Delta_1=[a_1,b_1]$, and L_{x_0} contains an infinite number of elements. Then for all $\varepsilon>0$ there exists N such that if $n>N$ and $(i,n)\in L_{x_0}$, then $\sup_{x\in\Delta\setminus\Delta_1} \varphi_i^n(x)<\varepsilon$.

Proof. Let $a_2\in(a_1,x_0)$, $b_2\in(x_0,b_1)$. Denote $h(x)=\bar{\delta}_{a_2,b_2}(x)$. From the H-continuoity of h it follows that $r(\Delta,1;\Phi_n(h),h)\xrightarrow[n\to\infty]{} 0$, whence for each $\varepsilon>0$ there exists N such that if $n>N$ then $\sup_{x\in\Delta\setminus\Delta_1} \Phi_n(h;x)<\varepsilon$ ($r(\Delta,1;f_1,f_2)$ is the Hausdorff distance with a parameter 1 between functions f_1 and f_2 in Δ and will be denoted further only by $r(f_1,f_2)$ for brevity).

Let integers i and n satisfy $n > N$ and $(i,n) \in L_{x_0}$. Then for all x in Δ $\varphi_i^n(x) \leq \sum_{x_{j,n} \in [a_2,b_2]} \varphi_j^n(x) = \Phi_n(h;x)$ and consequently $\sup_{x \in \Delta \setminus \Delta_1} \varphi_i^n(x) \leq \sup_{x \in \Delta \setminus \Delta_1} \Phi_n(h;x) < \varepsilon$. □

3. Counterexample construction. To prove the Theorem, we shall find a function f satisfying

(1) $r(\Phi_n(f),f) \xrightarrow[n\to\infty]{} 0$;

(2) $F(f) = I^2$,

where $F(f)$ denotes the completed graph of f.

To this end we construct a sequence of functions $f_0, f_1, \ldots$ such that $r(\Phi_n(f_n), I^2) \xrightarrow[n\to\infty]{} 0$ and the function f defined by $f(x) = \lim_{n\to\infty} f_n(x)$ satisfies the conditions (1) and (2).

For all integers j and k, $0 \leq j \leq 3^k$ denote

$$x_j^k = j.3^{-k} ,$$

$$A_j^k = [x_j^k + 3^{-k-2} , x_j^k + 2.3^{-k-2}] ,$$

$$C_j^k = [x_{j+1}^k - 2.3^{-k-2} , x_{j+1}^k - 3^{-k-2}] ,$$

$$D_j^k = [x_j^k + 3^{-k-2} , x_{j+1}^k - 3^{-k-2}] .$$

The sequence $\{f_n\}$ will be defined recursively. Suppose that functions $f_0, f_1, \ldots, f_n$ and integers $m_0 \leq m_1 \leq \ldots \leq m_n$ are determined such that if $1 \leq k \leq n$ and $i \geq m_1$ then

(3) $\inf_{x \in A_j^k} \Phi_i(f_k;x) \leq 3^{-l-1}$;

(4) $\sup_{x \in C_j^k} \Phi_i(f_k;x) \geq 1-3^{-l-1}$; $l=k-1,k$,

and if $i \geq m_{k-1}$ then

(5) $\Phi_i(f_k) = \Phi_i(f_{k-1})$.

We shall construct a function f_{n+1} and a number $m_{n+1} \geq m_n$ such that conditions (3), (4) and (5) are satisfied for k=n+1 and

$l=n, n+1$. In steps 1), 2), 3) to follow we determine 3 integers n_1, n_2 and n_3, required for the construction of f_{n+1}.

1) From Lemma 1 there exists $n_1 \geq m_n$ such that if $i > n_1$ then $\sup_{x \in D_j^{n+1}} \Phi_i(\underline{\delta}_{x_j^{n+1}, x_{j+1}^{n+1}}; x) \leq 3^{-n-1}$, $j=0,1, \ldots, 3^{n+1}-1$, which means that if functions h_1 and h_2 are equal in the interval $[x_j^{n+1}, x_{j+1}^{n+1}] \supset D_j^{n+1}$, then $\Phi_i(h_1)$ and $\Phi_i(h_2)$ differ in D_j^{n+1} by no more than 3^{-n-1}.

2) Let $X= \{x_{j,i} : 0 \leq j \leq i \leq n_1\}$,

$$L= \{(j,l) : x_{j,l} \in X \text{ and } L_{x_{j,l}} \text{ is a finite set}\}.$$

Since L itself is finite, $\max\{l : (j,l) \in L \text{ for some } j\}=\tilde{n}_2<\infty$. So if $l > \tilde{n}_2$, $0 \leq j \leq l$ and $x_{j,l} \in X$, then $L_{x_{j,l}}$ contains infinite number of elements.

Denote $\gamma=3^{-n-4}/(n_1+1)^2$, $L'= \{(j,l) : 0 \leq j \leq l,\ l \geq n_1,\ x_{j,l} \in X\}$ and $X_{j,l}= \{x \in \Delta : |x-x_{j,l}| > \gamma\}$, $(j,l) \in L'$. Since X contains a finite number of elements (at most $(n_1+1)(n_1+2)/2 \leq (n_1+1)^2$), then from Lemma 2 there exists $n_2 > \tilde{n}_2$ such that for <u>all</u> $(j,l) \in L'$

$$\sup_{x \in X_{j,l}} \varphi_j^l(x) \leq 3^{-n-1}/(n_1+1)^2.$$

Let $\tilde{X}= \bigcap_{(j,l) \in L'} X_{j,l}$. If f_1 and f_2 are functions satisfying $f_1(x)=f_2(x)$ for $x \in \Delta \setminus X$ and $l \geq n_2$, then from the choice of n_2 for all $x \in \tilde{X}$ $|\Phi_l(f_1;x)-\Phi_l(f_2;x)| \leq 3^{-n-1}$. Besides each subinterval of Δ with length 3^{-n-3} contains points from $\tilde{X}$.

3) From the convergence of $\{\Phi_n\}$ for the function $\bar{\delta}_{a,b}$, $0 \leq a < b \leq 1$, there exists $n_3 \geq n_2$ such that if

$$g_i^j(x)= \Phi_i(\bar{\delta}_{x_{9j+5}^{n+2}, x_{9j+6}^{n+2}})$$

and $i > n_3$, $0 \leq j \leq 3^n-1$, then

$$\inf_{x \in D_{3j+1}^{n+1}} g_i^j(x) \leq 3^{-n-1}; \tag{6}$$

(7) $$\sup_{x \in D^{n+1}_{3j+1}} g^j_i(x) \geq 1-3^{-n-1} .$$

Now we are ready to define f_{n+1}. Let $x=x_{k,i}$ and from $x_{k,i} =x_{k',i'}$ $i \leq i'$ follows. Then if $0 \leq j \leq 3^n-1$ let $f_n(x_{k,i})$ be equal to:

$f_n(x_{k,i})$, if $i \leq n_1$;

1, if $x_{k,i} \in [x^n_j+2.3^{-n-2}, x^n_j+3^{-n-1}]$, $i > n_3$;

1, if $x_{k,i} \in [x^n_j+3^{-n-1}+2.3^{-n-2}, x^n_j+2.3^{-n-1}]$, $i > n_1$;

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1, if $x_{k,i} \in [x^n_{j+1}-3^{-n-2}, x^n_{j+1}]$, $i > n_1$;

0, otherwise .

For all other x in Δ define $f(x)=0$.

Similarly as in Step 2) above, it can be shown that there exists N such that if $i > N$ and $0 \leq j \leq 3^{n+1}$

(8) $$\inf_{x \in A^{n+1}_j} \Phi_i(f_{n+1};x) \leq 3^{-n-2} ,$$

(9) $$\sup_{x \in C^{n+1}_j} \Phi_i(f_{n+1};x) \geq 1-3^{-n-2} .$$

Let $m_{n+1}=\max\{n_2, N\}$ and $m_n \leq i \leq m_{n+1}$. We shall prove that for $j \in [0, 3^n-1]$

(10) $$\inf_{x \in D^n_j} \Phi_i(f_{n+1};x) \leq 3^{-n} ,$$

(11) $$\sup_{x \in D^n_j} \Phi_i(f_{n+1};x) \geq 1-3^{-n} .$$

If $m_n \leq i \leq n_1$ then $f_{n+1}(x)=f_n(x)$ and the stronger inequalities (3) and (4) hold.

If $n_1 < i \leq n_3$ then the difference between $\Phi_i(f_{n+1};x)$ and $\Phi_i(f_n;x)$ in the intervals A^n_j and C^n_j as a result of the difference between f_n and f_{n+1} in the intervals $[x^n_l+3^{-n-1}, x^n_l+2.3^{-n-1}]$, $0 \leq l \leq 3^n-1$, does not exceed 3^{-n-1}, i.e. (10) and (11) are satisfied.

And finally if $n_3 < i \leq m_{n+1}$ then the difference between

$\Phi_i(f_{n+1};x)$ and $g_i^j(x)$ on the interval $D_{3j+1}^{n+1} \subset D_j^n$ as a result of the difference between f_{n+1} and g_i^j outside the interval $[x_{3j+1}^{n+1}, x_{3j+2}^{n+1}]$ and the choice of n_1 does not exceed 3^{-n-1}; as a result of their difference inside the interval $[x_{3j+1}^{n+1}, x_{3j+2}^{n+1}]$ and the choice of n_2 does not exceed 3^{-n-1}; or 2.3^{-n-1} together. Having in consideration the inequalities (6) and (7), the correctness of (10) and (11) follows.

The sequence $\{f_n\}$ will be correctly defined if f_0 and m_0 are determined. For f_0 we can take $\bar{\delta}_{0,2/3}$ and m_0 can be then found from the conditions (3) and (4).

Next we prove that the sequence constructed has the desired properties.

Let from $x_{k,i}=x_{k_1,i_1}$ $i \leq i_1$ follows, and let

$$m_{l-1} < i \leq m_l \quad . \tag{12}$$

Then for all $n \geq l$ $f_n(x)=f_l(x)$. Thus the sequence $\{f_n(x)\}$ is a sequence of a constant and the definition $f(x)=\lim_{n\to\infty} f_n(x)$ is correct. Furthermore $f(x)=f_l(x)$ and consequently $\Phi_i(f;x)=\Phi_i(f_l;x)$.

Obviously $F(f)=\{(x,y) : 0 \leq x, y \leq 1\} = I^2$. Then from (10) and (11)

$$r(\Phi_i(f),f)=r(\Phi_i(f_l),f) \leq 3^{-l+1},$$

where l is determined from (12), and

$$r(\Phi_i(f),f) \xrightarrow[i\to\infty]{} 0.$$

Thus the function f has the desired properties.

It is obvious that the result of the paper can be generalized to an arbitrary positive parameter α of the Hausdorff distance (instead of 1) and arbitrary interval (instead of Δ).

The speed of the convergence of $\{\Phi_n(f)\}$ depends of the properties of the oprators Φ_n, but we do not treat this problem here. It is an open problem to determine a function to replace f with highest possible speed of convergence.

References

1. Бл. Сендов. Хаусдорфовые приближения. Издательство Болгарской Академии Наук, София, 1979.

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