КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 298—299

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 298-299

SMOOTH FUNCTIONS AND ZERO TRACES IN SOBOLEV SPACE

Pavel Doktor

1. Notations. Let $\Omega\subset R_N$ be a bounded domain. Then $C^\infty(\bar\Omega)$ denotes, as usual, the set of infinitely differentiable functions defined in $\bar\Omega$, $C_0^\infty(\Omega)\subset C^\infty(\bar\Omega)$ is the subset of functions u with support supp u contained in Ω and $W^{1,p}(\Omega)$ ($1\leq p<+\infty$) stands for the usual Sobolev space.

We say that Ω has a continuous boundary, or a Lipschitzian boundary, iff the boundary of Ω can be locally described as a graph of a continuous function, or a Lipschitzian function, respectively, which locally devides R_N into the exterior and the interior of Ω (comp. [1,p.15]).

Let $\Gamma\subset\partial\Omega$ be a relatively open set (i.e. an open subset of the metric space $\partial\Omega$). We say that Γ has a continuous relative boundary $\partial^*\Gamma$ iff it posses the following property:

Let $x_0\in\partial^*\Gamma$ and let $U(x_0)$ be the neighbourhood of x_0 in which $\partial\Omega$ is described as a graph: $x_N = a(x_1, \dots x_{N-1})$, as above. Than the projection of $\partial^*\Gamma\cap U(x_0)$ on the hyperplane $x_N = 0$ is a part of the boundary ∂G of the projection G of $\Gamma\cap U(x_0)$ which is loccaly described by a continuous function (of N-2 variables) in the sense analogous to that described above.

2. Lemma. Let

$$C_M = \{x\in R_M : |x_i|<1\}$$

be an M-dimensional open cube, and let us denote $C_N^- = \{x\in C_N : x_N<0\}$. Let $G\subset\bar G\subset C_{N-1}$ be a domain with a continuous boundary, and let $\Gamma = G\times\{0\}\subset\partial C_N^-$. Let $u\in W^{1,p}(C_N^-)$ ($1\leq p<+\infty$) equals zero on Γ (in the sense of traces); let supp $u\subset\subset C_N$.

Then there exists a sequence u_n, $u_n\in C_0^\infty(C_N)$, such that supp $u_n\cap\bar\Gamma=\emptyset$ and $u_n\to u$ in $W^{1,p}(C_N^-)$.

Sketch of the proof:

(i) If $u = 0$ for $x_N = 0$ then we can extend u by zero to the whole C_N; then $u_\lambda(x) = u(x_1, \dots x_{N-1}, x_N + \lambda) \to u(x)$ in $W^{1,p}(C_N^-)$ if $\lambda \to 0^+$. Setting $u_{\lambda,h} = u_\lambda * \omega_h$ with a molifier ω_h of diameter h we easily obtain $u_{\lambda,h} \to 0$ for $\lambda = \frac{1}{n}$, $h = \frac{1}{2n}$, $n \to +\infty$, and, moreover, $\mathrm{supp}\, u_{\lambda,h} \subset C_N^-$.

(ii) If $u = 0$ in some neighbourhood of $\overline{\Gamma}$ then choosing $h = h(n)$ small enough we obtain that the sequence $u_{-1/n, h(n)}$ has the desired property.

(iii) General case can be reduced to the cases (i), (ii) via partition of unity as follows: We cover $\partial\Gamma$ by a finite number of open sets $U_i \subset \overline{U}_i \subset C_N$ ($i = 1, \dots r$) such that $\partial\Gamma \cap U_i$ is a graph (of a function of N-2 variables). We add open sets U_0, U_{r+1} in such a manner that $U_0, \dots U_{r+1}$ forms covering of supp u and then we consider functions $u_i = u\varphi_i$, with φ_i being a partition of unity. The function u_0 can be handled as in (i), and as for the function u_{r+1}, there can be applied (ii). The case $1 \le i \le r$ can be dealt with as (ii) after a suitable shift in the direction of the plane $x_N = 0$.

3. Theorem. Let $\Omega \subset R_N$ be a bounded set with a Lipschitzian boundary. Let $\Gamma \subset \partial\Omega$ be a relatively open set with a continuous boundary, and let $M_\Gamma = \{u \in W^{1,p}(\Omega): \text{trace } u = 0 \text{ on } \Gamma\}$ $(1 \le p < +\infty)$.

Then the set $\{u \in C^\infty(\Omega): \mathrm{supp}\, u \cap \Gamma = \emptyset\}$ is dense in M_Γ.

This theorem is a consequence of the lemma above (cf. [2]) and can be proved using a partition of unity and the Lipschitz mapping $y_i = c_i x_i$ ($i = 1, \dots N-1$), $y_N = c_N(x_N - a(x_1, \dots x_{N-1}))$ where a is a function (locally) describing the boundary $\partial\Omega$ and c_i are " normalising coefficients".

References

1. J.Nečas .Les méthodes directes en théorie des équations elliptiques, Academia, Prague, 1967.

2. P. Dokter. On the density of smooth functions in certain subspaces of Sobolev space. Comment. Math. Univ. Carolinae 14,4 (1973), 609-622 .

Department of Mathematics
Charles University, Prague
180 00 Prague , Sokolovská 83 CSSR