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ERROR ESTIMATES FOR A DISCRETE APPROXIMATION TO NONLINEAR OPTIMAL PROCESSES BY AVERAGED MODULI OF SMOOTHNESS

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1. Introduction. The averaged moduli of smoothness, originally defined by Sendov [1] , have been developed recently for approximations in differential equations with nonsmooth data.The aim of the present paper is to demonstrate an application of this techniques to optimal control.We consider a nonlinear and nonconvex optimal control problem with inequality state and control constraints.The finite difference approximation provided by the simplest Euler scheme is analysed.In contrast to related works, see Ermol'ev et al. [3] ,Vasil'ev [4] and Malanowski [5] , our approach does not require any continuity properties of the data and of the optimal control.We obtain a general estimate of the optimal value error by means of averaged moduli of smoothness.In such a way we extend and generalize results from the above mentioned papers.

2.Averaged moduli of smoothness. Given a vector-function $y: [0,1] \longrightarrow R^n$, $y(t) = (y_1(t),\dots,y_n(t))$ define the averaged modulus of smoothness as

$$\tau(y,\delta) = \int_0^1 \omega(y,t,\delta)dt , \qquad (1)$$

where ω is the local modulus of continuity of y,that is

(2) $$\omega(y,t,\delta) = \sup\left\{ \sum_{j=1}^{n} |\Delta^h(y_j(s))| \quad ; s, s+h \in \left[t-\frac{\delta}{2}, t+\frac{\delta}{2}\right]\right\},$$

$$\Delta^h(y_j(s)) = y_j(s+h) - y_j(s),\ j = 1,\dots,n .$$

It is known that if y is Riemann integrable then $\tau(y,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$, moreover $\tau(y,\delta) = O(\delta)$ when y has bounded variation. In the sequel we denote by RI the set of Riemann integrable functions and by BV the set of functions of bounded variation on the prescribed interval.

Let Y be a compact subset of R^n and the vector-function $f(y,t)$, $f\colon Y\times[0,1] \longrightarrow R^m$, be continuous with respect to y and measurable and bounded with respect to t. For fixed $y \in Y$ define the local modulus of continuity $\omega_y(f,t,\delta)$ according to (2). This is a function which is measurable and bounded with respect to $t \in [0,1]$ and continuous with respect to $y \in Y$. Thus, the following averaged modulus of smoothness is well-defined

(3) $$\overline{\tau}(f,\delta) = \int_0^1 \sup_{y\in Y} \omega_y(f,t,\delta)dt .$$

The most important for our purposes properties of $\overline{\tau}$ are:

(i) If $f(y,\cdot)$ is RI for all $y \in Y$ then $\overline{\tau}(f,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$;

(ii) If f is Lipschitz continuous with respect to y uniformly in t and has uniformly bounded variation in y, i.e. there exists $B > 0$ such that for any $0 \leq t_1 \dots t_k \leq 1$ and $y_1,\dots,y_{k-1} \in Y$ the following holds

$$\sup_k \sum_{i=1}^{k-1} |f(y_i,t_{i+1}) - f(y_i,t_i)| \leq B ,$$

then $\overline{\tau}(f,\delta) = O(\delta)$.

3. Problem. Consider the following optimal control problem denoted as (P_o):

$$\varphi(x(1)) \longrightarrow \min$$

subject to

$$\dot{x} = f(x,u,t) \ , \ x(0) = x^o \ , \tag{4}$$

$$\left.\begin{array}{ll} P(x(t)) \leqq 0 & (5) \\ Q(u(t)) \leqq 0 & (6) \end{array}\right\} \text{ for all } t \in [0,1],$$

where $x \in R^n$, $u \in R^m$, $P: R^n \longrightarrow R^p$, $Q: R^m \longrightarrow R^q$. Let us introduce the uniform grid $\{ih\}$, $t_i = ih$, $i = 0,\ldots,N$, $h = 1/N$. The discrete approximation (P_N) to (P_o) is analysed, namely

$$\varphi(x_N) \longrightarrow \min$$

subject to

$$x_{i+1} = x_i + hf(x_i,u_i,t_i), \ i = 0,\ldots,N-1, \tag{7}$$

$$x_o = x^o,$$

$$P(x_i) \leqq 0 \quad , \ i = 1,\ldots,N, \tag{8}$$

$$Q(u_i) \leqq 0 \quad , \ i = 0,\ldots,N-1. \tag{9}$$

In the next section we present an estimate of the distance between the optimal value $\hat{\varphi}$ of (P_o) and the optimal value $\hat{\varphi}_N$ of the approximate problem (P_N). This result is obtained on the following assumptions:

1^o. The function φ is Lipschitz continuous, f is differentiable with respect to (x,u) for every fixed $t \in [0,1]$; f and the derivatives $\partial f/\partial x$, $\partial f/\partial u$ are continuous in (x,u) and measurable and bounded in t .

2^o. $P_j(x^o) < 0$ for all $j = 1,\ldots,p$; P and Q are C^2; the set $\{u \in R^m, \ Q(u) \leqq 0\}$ is contained in a ball $U \subset R^m$; for every measurable control u, $u(t) \in U$, there exists unique solution x of the Cauchy problem (4) such that $x(t) \in X$, where X is fixed ball in R^n.

Let ε be a scalar parameter.Define the family of problems (P_ε): minimize $\varphi(x(1))$ subject to (4),(6) and

$$(10) \qquad P_j(x(t)) \leqq \varepsilon \quad , j = 1,\ldots,p.$$

We use this embedding to formulate the next assumption:

3^o. There exist $\varepsilon_0 > 0$ and a function $\sigma(h) \geqq 0$ such that for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ the problem (P_ε) has a solution $\hat{u}_\varepsilon$ so that $\tau(\hat{u}_\varepsilon, h) \leqq \sigma(h)$.

The following condition is a strengthened version of Hager's regularity condition, see [6] :

4^o. There exist $\delta_1 > 0$ and $\Gamma > 0$ such that for all $\delta \in (0, \delta_1)$ and for all $x \in X$, $u \in U$, and $t \in [0,1]$ the following inequality holds

$$\left| \left[\frac{\partial f}{\partial u}^T(x,u,t) P_x^\delta(x)^T ,\ Q_u^\delta(u)^T \right] z \right| \geqq \Gamma |z|$$

for every z, where P_x^δ and Q_u^δ denote those subvectors of the gradients of P and Q which correspond to δ-binding constraints at x and u respectively, i.e. $|P_i(x)| \leqq \delta$, $|Q_j(u)| \leqq \delta$.

<u>4.Result.</u> Let $\tau(f,h)$ be the averaged modulus of smoothness defined in (3) of the function $f(y.t)$, $y = (x,u) \in X \times U$, $t \in [0,1]$, and $\sigma(h)$ be as in 3^o.

Theorem 1. Let 1^o-4^o hold.Then there exists a constant $c > 0$ such that if $\hat{\varphi}$ and $\hat{\varphi}_N$ are the optimal values of the problems (P_0) and (P_N) respectively, then

$$|\hat{\varphi}_N - \hat{\varphi}| \leqq c(h + \sigma(h) + \tau(f,h)).$$

The proof id based on the following lemmas:

Lemma 1. If 1^o-4^o hold there exists $\varepsilon_1 > 0$ and a constant $c_1 > 0$ such that if $\hat{\varphi}_\varepsilon$ is the optimal value of (P_ε) then

$$|\hat{\varphi}_\varepsilon - \hat{\varphi}| \leqq c_1 \varepsilon$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.

Lemma 2. Let the feasible control u be fixed, x be the corresponding trajectory according to (4) and x_i^N solve (7) for $u_i = u(t_i)$. If 1^o and 2^o hold then there exists a constant $c_2 > 0$, independent of u and N such that

$$|x(t_i) - x_i^N| \leqq c_2(h + \tau(u,h) + \overline{\tau}(f,h)),\ i = 1,\dots,N$$

Lemma 3. Let u_i^N be a discrete feasible control and x_i^N be the correponding trajectory according to (7). Let u^N be a step function, $u^N(t_i) = u_i^N$ and x^N be the corresponding solution of (4). If 1^o and 2^o hold then there exists a constant c_3 independent of u_i^N such that

$$|x_i^N - x(t_i)| \leqq c_3(h + \overline{\tau}(f,h)),\ i = 1,\dots,N.$$

Remarks. If the state constraints (5) are vacuous then the conditions 3^o and 4^o in Theorem 1 can be dropped. When the function $f(x,u,\cdot)$ and the optimal control $\hat{u}_\varepsilon$ for (P_ε) are RI then we get convergence $\hat{\varphi}_N \longrightarrow \hat{\varphi}$ as $N \longrightarrow \infty$. In order to estimate the rate of convergence one needs, however, more conditions. To be specific, consider the examplary problem

$$\varphi(x(1)) \longrightarrow \min$$

subject to

$$\dot{x} = \sum_{j=1}^{k} a_j(t) b_j(x,u),\ x(0) = x^0,$$

(5) and (6),

where 1^o-4^o hold, a_j are BV and b_j are Lipschitz continuous. Suppose additionally that the Hamiltonian in Pontryagin maximum principle is in the normal form and it is strongly concave with respect to u uniformly in x and t. Then, by repeating the argument in Dontchev[7], p.56 one can prove that the optimal control has bounded variation. Thus, by the property (ii) of $\tau(\hat{u},h)$ and Theorem 1 we conclude that

$$|\hat{\varphi}_N - \hat{\varphi}| = O(h).$$

References

1. Бл.Сендов. Аппроксимация относительно хаусдорфова расстояния, Диссертация, Москва 1967.

2. Бл.Сендов, В.Попов. Усреднени модули на гладкост. Издателство на БАН, София 1983.

3. Ю.М.Ермольев, В.П.Гуленко, Т.И.Царенко. Конечно-разностный метод в задачах оптимального управления. Наукова думка, Киев 1978.

4. Ф.П.Васильев. Методы решения экстремальных задач. Наука, Москва, 1981.

5. K.Malanowski. On convergence of finite difference approximation to optimal control problems for system with control appearing linearly. Archiwum Automatyki i Telemechaniki. t.24, 1979, p.155-171.

6. W.W.Hager. Lipschitz continuity for constrained processes. SIAM J.Control and Optimization. Vol.17, 1979, 321-338.

7. A.L.Dontchev. Perturbations, approximations and sensitivity analysis of optimal control systems. Lecture Notes in Control and Inf. Sciences, vol.52, Springer, Berlin-Heidelberg- New York-Tokyo, 1983.

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