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THE CONVERGENCE OF THE DIFFERENCE SCHEME OF THE FINITE ELEMENT METHOD FOR THE POISSON EQUATION IN A DISCRETE L_p-NORM

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Investigation of the accuracy of discrete methods for the Poisson equation in L_p metrics, $p \neq 2$ is of interest both for the general theory and for comparing the application possibilities of the finite element method (FEM) and finite difference method, widely used for solving differential equations.

The main purpose of this article is to study the accuracy of a scheme obtained by FEM in L_p-norm, $p \neq 2$. The convergence in L_p metrics of a finite difference scheme is obtained in [3].

1. We consider the Dirichlet problem for the Poisson equation in the rectangle $\Omega = \{x = (x_1,x_2),\ |x_i|<\pi,\ i = 1,2\}$ with boundary Γ

(1) $$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = f(x),\ x \in \Omega$$

(2) $$u(x) = 0,\ x \in \Gamma .$$

We suppose that the solution $u(x)$ of (1), (2) belongs to the Sobolev class $W_p^m(\Omega)$, $m = 1,2,3,4$, $1 < p < \infty$.

2. In the domain Ω we introduce a mesh $\bar\omega = \{x_k = ih_k,\ i = 0,1,\dots,N_k,\ h_k = \pi N_k^{-1},\ k = 1,2\}$. Denote by $\omega = \bar\omega \cap \Omega$ the set of internal nodes of $\bar\omega$ and by $\gamma = \bar\omega \setminus \omega$ the set of boundary nodes. We will use the conventions set forth in [6] for functions given on $\bar\omega$ and the operations on them. In what follows we denote by C a generic positive constant which does not depend on $|h|$, p and some functions. It will be clear from the context of which functions the

constant is independent.

We introduce [6] two triangulations of Ω. In the case of the "left" triangulation any partition is decomposed into two parts by the diagonal from the upper left corner to the lower right one. Similarly the "right" triangulation is defined by the diagonals connecting the other two vertices of any partition. In a standard way we introduce basic functions $\varphi_{ij}^{(l)}$ and $\varphi_{ij}^{(r)}$ related to the "left" and "right" triangulation resp.

The basic functions of the studied FEM scheme are defined by the formula $\varphi_{ij} = \frac{1}{2}(\varphi_{ij}^{(l)} + \varphi_{ij}^{(r)})$. The approximate solution $u_h(x)$ of (1), (2) can be given in the form

$$u_h(x) = \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} y_{ij}\varphi_{ij}(x).$$

The unknown coefficients y_{ij} of $u_h(x)$ satisfy the discrete problem

(3) $\quad \Lambda y = y_{\bar{x}_1 x_1} + y_{\bar{x}_2 x_2} = (f, \varphi_{ij})/h_1 h_2, \; x = (ih_1, jh_2) \in \omega$

(4) $\quad y(x) = 0, \; x \in \gamma.$

The following estimate is valid by standard arguments of the FEM [2]:

$$\|u - u_h\|_{L_2(\Omega)} \leq C|h|^m |u|_{W_2^m(\Omega)}, \; m = 1,2.$$

Our purpose is to generalize this result in the discrete L_p-norm, $1 < p < \infty$.

3. We shall examine the accuracy of the scheme (3), (4). We make use of the embedding theorem $W_p^m(\Omega) \to C(\Omega)$, $mp > 2$. Hence for $p > \max(\frac{2}{m}, 1)$ we can compare the exact and the approximate solutions in the knots of ω. The mesh function $z = y - u$ characterizes the difference scheme error. After the substitution $y = z + u$ into (3) we obtain that the error z satisfies the following problem:

(5) $\quad \Lambda z = \psi, \; x \in \omega,$

(6) $\quad z(x) = 0, \; x \in \gamma.$

The function $\psi = (f, \varphi_{ij})/h_1 h_2 - \Lambda u$ is the truncation error and can be presented in the form:

$$\psi = \eta^{(1)}_{\bar{x}_1 x_1} + \eta^{(2)}_{\bar{x}_2 x_2} + \eta^{(3)}_{\bar{x}_1 \bar{x}_2},$$

where the functions $\eta^{(i)}(x)$, $i = 1,2,3$ are given by the formula:

$$\eta^{(1)}(x_1,x_2) = \frac{1}{2}\int_{-1}^{1} u(x_1,x_2+ sh_2)ds - u(x_1,x_2),$$

$$\eta^{(2)}(x_1,x_2) = \frac{1}{2}\int_{-1}^{1} u(x_1+sh_1,x_2)ds - u(x_1,x_2),$$

$$\eta^{(3)}(x_1,x_2) = \frac{1}{2}\left(\frac{h_1}{h_2} + \frac{h_2}{h_1}\right)\int_{0}^{1}\Big[u(x_1+ sh_1,x_2- sh_2+ h_2) - u(x_1+ sh_1,x_2+ sh_2)\Big]ds.$$

Next we prove an a priori estimate in $L_p(\omega)$ for the solution z of (5), (6). This estimate is the most important step in the proof of the main result.

<u>Lemma 1</u>. Let $s = 0,1,2$, $p > \max(1,\frac{2}{m})$. Then the function $z(x)$, $x \in \bar{\omega}$, satisfies the inequalities:

$$\|z\|_{W_p^s(\omega)} \leq C\left(\frac{p^2}{p-1}\right)^4 \sum_{j=1}^{3} \|\eta^{(j)}\|_{L_p(\omega)}, \quad s = 0$$

$$\|z\|_{W_p^s(\omega)} \leq C\left(\frac{p^2}{p-1}\right)^4 \Big\{\|\eta^{(1)}_{\bar{x}_1}\|_{L_p(\omega)} + \|\eta^{(2)}_{\bar{x}_2}\|_{L_p(\omega)} + \|\eta^{(3)}_{\bar{x}_1}\|_{L_p(\omega)} + \|\eta^{(3)}_{\bar{x}_2}\|_{L_p(\omega)}\Big\}, \quad s = 1$$

$$\|z\|_{W_p^s(\omega)} \leq C\left(\frac{p^2}{p-1}\right)^4 \Big\{\|\eta^{(1)}_{\bar{x}_1 x_1}\|_{L_p(\omega)} + \|\eta^{(2)}_{\bar{x}_2 x_2}\|_{L_p(\omega)} + \|\eta^{(3)}_{\bar{x}_1 \bar{x}_2}\|_{L_p(\omega)}\Big\}, \quad s = 2.$$

The proof of this lemma is based on the method of multipliers in the discrete space $L_p(\omega)$ [5].

Lemma 2. Let the solution u(x) of the problem (1), (2) belong to the class $W_p^m(\Omega)$, $mp > 2$. Then there exists a constant C such that

$$\|\eta^{(j)}\|_{L_p(\omega)} \leqq C|h|^m |u|_{W_p^m(\Omega)}, \quad m = 1,2,\ j = 1,2,3,\ p > \max(\tfrac{2}{m},1)$$

$$\|\eta^{(j)}_{\bar{x}_j}\|_{L_p(\omega)} \leqq C|h|^{m-1} |u|_{W_p^m(\Omega)}, \quad m = 2,3,\ j = 1,2,\ p > 1$$

$$\|\eta^{(3)}_{\bar{x}_j}\|_{L_p(\omega)} \leqq C|h|^{m-1} |u|_{W_p^m(\Omega)}, \quad m = 2,3,\ j = 1,2,\ p > 1$$

$$\|\eta^{(i)}_{\bar{x}_i x_i}\|_{L_p(\omega)} \leqq C|h|^{m-2} |u|_{W_p^m(\Omega)}, \quad m = 3,4,\ i = 1,2,\ p > 1$$

$$\|\eta^{(3)}_{\bar{x}_1 \bar{x}_2}\|_{L_p(\omega)} \leqq C|h|^{m-2} |u|_{W_p^m(\Omega)}, \quad m = 3,4,\ p > 1.$$

The proof of Lemma 2 is based on the Bramble - Hilbert lemma [1].

The main result of this article is the next theorem.

Theorem 1. Let the solution u(x) of the boundary problem (1), (2) be from the Sobolev class $W_p^m(\Omega)$, $m = 1,2,3,4$, $p > \max(1,\frac{2}{m})$ and let $s = 0,1,2$, $\max(0,m-2) \leqq s < m$. Then the following estimate holds

$$\|y - u\|_{W_p^s(\omega)} \leqq C\left(\frac{p^2}{p-1}\right)^4 |h|^{m-s} |u|_{W_p^m(\Omega)}.$$

The proof of this theorem follows directly from Lemma 1 and Lemma 2.

This theorem provides comprehensive information about the accuracy of the scheme (3), (4). We remark that we can estimate the values $|u|_{W_p^m(\Omega)}$ in terms of the corresponding norms of f(x), i.e. directly in terms of the input data of the problem.

4. Theorem 1 is valid for $p > 2$ if $u(x) \in W_p^1(\Omega)$. It is of interest to estimate the accuracy for values of p from the interval (1,2]. For such p we are not able to determine the solution pointwize. Thus we can examine the closeness of the mesh function y to some arbitrary average value of u(x).

We introduce a local averaging $\tilde{u}$ of u:

$$\tilde{u}(x) = \begin{cases} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} u(x_1 + s_1 h_1, x_2 + s_2 h_2) ds_1 ds_2, & x \in \omega \\ 0, & x \in \gamma \end{cases}$$

and compare the approximate solution y with $\tilde{u}$. Using arguments in the spirit of Lemma 1 and Lemma 2 we obtain the following result:

Theorem 2. Let the solution u(x) of (1), (2) be from the class $W_p^1(\Omega)$. Then there exists a constant C such that

$$\left\| y - \tilde{u} \right\|_{L_p(\omega)} \leq C \, |h| \left(\frac{p^2}{p-1} \right)^4 |u|_{W_p^1(\Omega)} .$$

5. Generalizations and remarks.

(i) The method of investigation is also applicable to elliptic problems with constant coefficients

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - qu = f(x), \quad x \in \Omega$$

$$u(x) = 0, \quad x \in \Gamma .$$

The rate of convergence is the same as in Theorem 1 and Theorem 2.

(ii) Using interpolated theorems for the discrete space $W_p^s(\omega)$, the result of Theorem 1 can be extended for noninteger s such that $\max(0, m - 2) \leq s < m$.

(iii) The complete proof of Theorem 1 in the cases m = 1 and m = 2 is given in [4].

6. In this way the scheme (3), (4) obtained by FEM correctly reflects the differential properties of the exact solution u(x) of (1), (2). It has the same characteristics with respect to accuracy estimates as the scheme [3] obtained by the finite difference method.

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